BUILDING RESEARCH JOURNAL

VOLUME 60, 2012 NUMBER 3 – 4

MESH-FREE FORMULATIONS FOR SOLUTION OF BENDING PROBLEMS FOR THIN ELASTIC PLATES WITH VARIABLE BENDING STIFFNESS. PART I: MATHEMATICAL FORMULATION

VLADIMIR SLADEK, JAN SLADEK AND LADISLAV SATOR

The first part of the paper is devoted to a mathematical formulation of the plate bending problems within the classical Kirchhoff-Love theory for bending of thin elastic plates. The functional gradations of material coefficients due to the in-plane and transversal continuous variation of the Young modulus are considered separately. The plate thickness is allowed to be variable with respect to the in-plane coordinates. The effects associated with the transversal gradation are discussed. The decomposed formulation is proposed in order to decrease the order of the derivatives in the governing equations and the boundary densities playing role in boundary conditions. In order to get the exact solutions usable as benchmark solutions in numerical tests, we developed also the formulation for circular plates with central circular hole. Finally, the exact solutions are derived for the angular symmetric plate bending problems with variable bending stiffness.

Keywords: Kirchhoff-Love theory, variable thickness, functionally graded materials, governing equations, exact solutions, decomposed formulation, plane stress state, coupling effects

1. Introduction

Since 1984, when a novel materials concept was first considered in Japan (Koizumi, 1993; Yamanouchi, Koizumi, Hirai, Shiota eds., 1990), the functionally graded materials (FGM) have gained considerable attention as a potential structural material for the future. FGMs are composite materials, microscopically inhomogeneous, with gradual variations in composition and structure over the volume, resulting in corresponding smooth and continuous changes in macroscopic properties of materials. Using various processing approaches, one can fabricate FGMs designed for specific function and applications.

In conventional laminated composite structures, homogeneous elastic laminae are bonded together to obtain enhanced mechanical properties. However, the abrupt change in material properties across the interface between different materials can result in large stress discontinuities leading to delamination. This undesirable effect can be overcome by replacing the laminated composite

Institute of Construction and Architecture, Slovak Academy of Sciences, Dubravska cesta 9, 845 03 Bratislava 45, Slovak Republic, e-mail: vladimir.sladek@savba.sk, jan.sladek@savba.sk, ladislav.sator@savba.sk

structure with the FGM structure, where the stress distributions are smooth. A great attention has been paid to FG plates using various computational methods for numerical analyses (see e.g. Zenkour, 2005).

Plates are defined as plane structural elements with a small thickness compared to the planar dimensions. The typical thickness to width ratio of a plate structure, h / L, is less than 0.1. The advantage of a plate theory consists in reduction of the full three-dimensional solid mechanics problem to a two-dimensional problem. Of the numerous plate theories that have been developed since the late 19th century, three are widely accepted and used in engineering. These are:

- the Kirchhoff–Love theory of plates – KLT (classical plate theory) (Timoshenko, Woinowsky-Krieger, 1959),
- the Mindlin–Reissner theory of plates (first-order shear deformation plate theory) (Reddy, 1997),
- the shear deformation plate theories (SDPT) including the first order SDPT (FSDPT), the third order SDPT (TSDPT) and other higher order SDPT (Reddy, 1997; Zenkour, 2005).

Recall that the KLT does not involve the shear deformations and the dimensionless deflections $w^* = w / h$ corresponding to unite dimensionless load $q^*=(L^4 / Dh)q$ are independent on the thickness of the plate. Apparently, such a theory cannot be applicable to plates with a rather wide range of the thickness to wideness ratio. Nevertheless, there are several serious reasons why to pay attention to the KLT. Firstly, the KLT gives reasonably accurate results for sufficiently thin plates and the computations are significantly simpler than in the case of SDPT. Secondly, there are interesting phenomena associated with functional gradation of material of plates. The investigation of such phenomena within the KLT is one of the main objections of this paper. Recall that most papers devoted to FG plates in literature concerns the transversal gradation of material coefficients. The spatial gradation of material coefficients and/or the plate thickness with respect to the in-plane coordinates does not bring to any new phenomena as compared with the homogeneous plate, but the governing equation is much more complicated because it becomes the PDE with variable coefficients in contrast to the PDE with constant coefficients. On the other hand, the gradation of the Young modulus across the plate thickness gives rise to the coupling between the plate deflection and the in-plane deformations even in the case of pure transversal loading of the plate. Although such a coupling is well discussed within the SDPT, it seems it is underestimated in the KLT, where in the case of homogeneous plate there is no coupling effect. Maybe this is misleading to reduce the effect of transversal gradation of the Young modulus to bearing the variable Young modulus only in the definition of the stress couples with omitting the actual coupling between the deflection and in-plane deformations. Note that the plate bending problem can be solved separately from the in-plane deformations also in the case of transversal gradation of the Young modulus, but the bending stiffness is modified not only because of the functional dependence of the Young modulus but also owing to the coupling effect. Furthermore, in a homogeneous plate subjected to only transversal loading, the average transversal stresses vanish which is compatible with the plane-stress assumption in the generalized sense (Barber, 2010; Lurie, 2005). In the FGM plates with transversal gradation of the Young modulus, the average transversal stresses are different from zero. Hence, the assumption of the plane-stress state could be questionable in such a case. This question will be discussed in details. In any case, the governing equation for plate deflections in the KLT is the PDE of the 4^{th} order either with variable or constant coefficients. The boundary conditions are expressed in terms of the deflections and their derivatives which could be also of the 3^{rd} order. Recall that the high order derivatives are inappropriate from the point of view of numerical methods because of increasing inaccuracy with increasing their order. The computational efficiency suffers also from the evaluation of high-order derivatives especially in case of meshless approximations.

Another point, which will be paid attention in this paper, is the decreasing of the order of the derivatives involved in the governing equation and the boundary conditions of the plate bending problems in the KLT. The decomposed formulation is proposed with introducing a new field variable in addition to the deflection field. Then, the governing equations are coupled 2nd order PDE. For the sake of simplicity and possibility to find the exact solution, we deal with the circular plate problems exhibiting the angular symmetry. Then, the governing equations become the ordinary differential equations and the derivatives of the field variables in the relevant boundary quantities do not exceed the first order.

Finally, we present the derivation of the exact solutions for plates with power-law graded bending stiffness as well as for FGM plates with power-law graded Young's modulus across the plate thickness. These solutions can be used as benchmark solutions in the second part of the paper for tests of accuracy and convergence study of the proposed method for numerical solution.

2. Governing equations in the Kirchhoff-Love theory

Let us consider a straight plate structure occupying the 3D domain

$$V=\{\forall(x_1,x_2,x_3)\in\mathbb{R}^3;\ \mathbf{x}=(x_1,x_2)\in\Omega,\ x_3\in[-h/2,\ h/2]\}=\Omega\times[-h/2,\ h/2]$$

According to the Kirchhoff hypothesis the 3D displacement field is expressed as

$$\tilde{u}_i(\mathbf{x},x_3)=\delta_{i\alpha}\left[u_\alpha(\mathbf{x})-x_3 w_{,\alpha}(\mathbf{x})\right]+\delta_{i3}w(\mathbf{x}) \quad , \tag{1}$$

where $u_\alpha(\mathbf{x})$ are the in-plane displacements of the mid-surface Ω with $\alpha\in\{1,2\}$, and $w(\mathbf{x})$ is the displacement of the mid-surface in the x_3-direction. The Einstein summation convention of summing on repeated indices is used. If φ_α is the angle of rotation of the normal to the mid-surface around x_a-axis, then $\varphi_\alpha=w_{,\alpha}$ in the Kirchhoff-Love theory (Timoshenko, Woinowsky-Krieger, 1959) where the straight lines normal to the mid-surface remain straight and normal to the mid-surface after deformation in contrast to the Mindlin-Reissner theory (Reddy, 1997), where the normal to the mid-surface remains straight but not necessarily perpendicular to the mid-surface.

Assuming small deformations, we obtain the strains corresponding to displacements given by Eq. (1) as

$$\begin{aligned}
&\tilde{\varepsilon}_{\alpha\beta}(\mathbf{x},x_3)=\frac{1}{2}\left(u_{\alpha,\beta}(\mathbf{x})+u_{\beta,\alpha}(\mathbf{x})\right)-x_3 w_{,\alpha\beta}(\mathbf{x})\\
&\tilde{\varepsilon}_{\alpha 3}(\mathbf{x},x_3)=\frac{1}{2}\left(w_{,\alpha}(\mathbf{x})-w_{,\alpha}(\mathbf{x})\right)\equiv 0\,,\quad \tilde{\varepsilon}_{33}(\mathbf{x},x_3)=w_{,3}(\mathbf{x})\equiv 0\ .
\end{aligned} \tag{2}$$

Thus, the shear strain $\tilde{\varepsilon}_{\alpha 3}$ is vanishing in the Kirchhoff-Love theory in contrast to the shear deformation plate theories, where it is finite and constant across the plate thickness in the Mindlin-Reissner theory. In view of strains (2) and the Hooke law, the 3D elastic stresses in the plate structure are given as

$$\tilde{\sigma}_{\alpha\beta}(\mathbf{x},x_3)=\frac{E}{1-\nu^2}\frac{1-\nu}{H}\tau_{\alpha\beta}(\mathbf{x})-\frac{E}{1-\nu^2}\frac{1-\nu}{H}x_3\mu_{\alpha\beta}(\mathbf{x}), \quad ,$$
$$\tilde{\sigma}_{\alpha 3}(\mathbf{x},x_3)=\tilde{\sigma}_{3\alpha}(\mathbf{x},x_3)=0, \qquad \tilde{\sigma}_{33}(\mathbf{x},x_3)=\frac{E\nu}{1-\nu^2}\frac{1-\nu}{H}\left(u_{\alpha,\alpha}(\mathbf{x})-x_3 w_{,\alpha\alpha}(\mathbf{x})\right), \tag{3}$$

with

$$\tau_{\alpha\beta}:=\frac{H}{2}\left(u_{\alpha,\beta}(\mathbf{x})+u_{\beta,\alpha}(\mathbf{x})\right)+\nu\delta_{\alpha\beta}u_{\gamma,\gamma}(\mathbf{x}), \quad \mu_{\alpha\beta}:=Hw_{,\alpha\beta}(\mathbf{x})+\nu\delta_{\alpha\beta}w_{,\gamma\gamma}(\mathbf{x}) \tag{4}$$

where E and ν is the Young modulus and the Poisson ratio, respectively. According to Eq. (2), we are dealing with a plane-strain problem, hence and consistently with the 3D elasticity, we should take $H = 1 - 2\nu$.

Recall that the only dependence on x_3 is determined by the assumption (1) leading to vanishing transversal (and also shear) deformations across the plate thickness. Thus, there is no effect of transversal loading on the transversal deformations but only on the bending of the plate (Dym, Shames, 2013). Since $h \ll L$ (where h and L are the thickness and a characteristic width of the plate, respectively), the variations on $x_3 \in [-h/2, h/2]$ can be treated by using the average stresses and stress couples across the thickness of the plate

$$T_{ij}(\mathbf{x}):=\int_{-h/2}^{h/2}\tilde{\sigma}_{ij}(\mathbf{x},x_3)dx_3, \qquad M_{\alpha\beta}(\mathbf{x}):=\int_{-h/2}^{h/2}x_3\tilde{\sigma}_{\alpha\beta}(\mathbf{x},x_3)dx_3 \quad , \tag{5}$$

Recall that the in-plane displacements field α $u_\alpha(\mathbf{x})$ and the deflection field $w(\mathbf{x})$ are the primary fields, while their gradients $u_{\alpha,\beta}(\mathbf{x})$, $w_{,\alpha\beta}(\mathbf{x})$ [or the deformation fields $\tau_{\alpha\beta}(\mathbf{x})$ and $\mu_{\alpha\beta}(\mathbf{x})$] are the secondary fields and the conjugated field in the semi-integral formulation are $T_{ij}(\mathbf{x})$ and $M_{\alpha\beta}(\mathbf{x})$.
The principle of virtual work

$$\int_\Omega\left(\int_{-h/2}^{h/2}\tilde{\sigma}_{ij}\delta\tilde{\varepsilon}_{ij}dx_3\right)d\Omega-\int_\Omega q\delta w d\Omega=0$$

yields

$$\int_\Omega\left(T_{\alpha\beta}\delta u_{\alpha,\beta}-M_{\alpha\beta}\delta w_{,\alpha\beta}\right)d\Omega-\int_\Omega q\delta w d\Omega=0,$$

hence

$$\int_{\partial\Omega}\left(n_\beta T_{\alpha\beta}\delta u_\alpha-M\delta\left(\frac{\partial w}{\partial\mathbf{n}}\right)+V\delta w\right)d\Gamma-\int_\Omega\left[T_{\alpha\beta,\beta}\delta u_\alpha+\left(M_{\alpha\beta,\alpha\beta}+q\right)\delta w\right]d\Omega - \Sigma[\![T]\!]\delta w=0 \tag{6}$$

where M, T and V stand for the bending moment, twisting moment and generalized shear force defined as

$$M := n_\alpha n_\beta M_{\alpha\beta}, \quad V := n_\alpha M_{\alpha\beta,\beta} + \frac{\partial T}{\partial \mathbf{t}}, \quad T := t_\alpha n_\beta M_{\alpha\beta}$$

The last term in Eq. (6) is the sum of the possible jumps of the twisting moments on the closed boundary edge of the plate.

In view of (6), the governing equations in the semi-integral formulation are given as

$$T_{\alpha\beta,\beta} = 0, \tag{7}$$

$$M_{\alpha\beta,\alpha\beta} = -q \tag{8}$$

and the boundary conditions should obey the following equations at each boundary point

$$n_\beta T_{\alpha\beta} \delta u_\alpha = 0, \quad M\delta\left(\frac{\partial w}{\partial \mathbf{n}}\right) = 0, \ \left(V - \Sigma[\![T]\!]\right)\delta w = 0$$

hence, we can distinguish the following boundary conditions

$$\begin{aligned} &n_\beta(\mathbf{x}) T_{\alpha\beta}(\mathbf{x}) = 0 \ \text{ at } \ \mathbf{x} \in \partial\Omega_T, \\ &\text{or } u_\alpha(\mathbf{x}) = u_\alpha^{prescr}(\mathbf{x}) \ \text{ at } \ \mathbf{x} \in \partial\Omega_u, \ \text{ with } \ \partial\Omega_T \cup \partial\Omega_u = \partial\Omega \end{aligned} \tag{9}$$

$$\begin{aligned} &M(\mathbf{x}) = 0 \text{ at } \ \mathbf{x} \in \partial\Omega_M \ \text{ (on SSE or FE), or } \frac{\partial w}{\partial \mathbf{n}}(\mathbf{x}) = 0 \ \text{ at } \ \mathbf{x} \in \partial\Omega_{\partial w} \ \text{ (on CE)} \\ &V(\mathbf{x}) - \Sigma T(\mathbf{x}) = 0 \ \text{ at } \ \mathbf{x} \in \partial\Omega_V \ \text{ (on FE), or } w(\mathbf{x}) = 0 \ \text{ at } \ \mathbf{x} \in \partial\Omega_w \ \text{ (on CE or SSE)}, \end{aligned} \tag{10}$$

where (CE), (SSE) and (FE) stand for the clamped, simply supported and free edge, respectively. Note that $\partial\Omega_{CE} = \partial\Omega_w \cap \partial\Omega_{\partial w}$, $\partial\Omega_{SSE} = \partial\Omega_w \cap \partial\Omega_M$, $\partial\Omega_{FE} = \partial\Omega_M \cap \partial\Omega_V$, with $\partial\Omega = \partial\Omega_{CE} \cup \partial\Omega_{SSE} \cup \partial\Omega_{FE}$.

Now, assuming the material coefficients to be independent on x_3, we obtain from (3) – (5)

$$T_{\alpha\beta}(\mathbf{x}) = \frac{E(\mathbf{x})h}{1-\nu^2}\frac{1-\nu}{H}\tau_{\alpha\beta}(\mathbf{x}), \quad T_{\alpha 3}(\mathbf{x}) \equiv 0, \quad T_{33}(\mathbf{x}) = \frac{E(\mathbf{x})h}{1-\nu^2}\frac{1-\nu}{H}\nu u_{\alpha,\alpha}(\mathbf{x}), \tag{11}$$

$$M_{\alpha\beta}(\mathbf{x}) = -\frac{1-\nu}{H} D(\mathbf{x}) \mu_{\alpha\beta}(\mathbf{x}), \tag{12}$$

where

$$D(\mathbf{x}) := \frac{E(\mathbf{x})h^3(\mathbf{x})}{12(1-\nu^2)} \tag{13}$$

is the bending stiffness.

Note that only $T_{\alpha\beta}$ and $M_{\alpha\beta}$ occur in the governing equations (7) and (8), while T_{33} and $T_{\alpha 3}$ are irrelevant for solution of the considered bending problem. Really, T_{33} and $T_{\alpha 3}$ do not affect the energetic balance in the principle of virtual work (6) because $\tilde{\sigma}_{33}\delta\tilde{\varepsilon}_{33} = 0$ and $\tilde{\sigma}_{\alpha 3}\delta\tilde{\varepsilon}_{\alpha 3} = 0$, since $\tilde{\varepsilon}_{33} = const(=0)$ and $\tilde{\varepsilon}_{\alpha 3} = const(=0)$. Thus, the energetic balance as well as the governing equations will not be influenced if we take $\tilde{\sigma}_{33} = 0$ and $\delta\tilde{\varepsilon}_{33}$ arbitrary. In other words, the fields $\tilde{\sigma}_{33}$ and $\tilde{\varepsilon}_{33}$ do not affect the solution of the bending problem and we can take $\tilde{\sigma}_{33} = 0$, $\tilde{\varepsilon}_{33} = -\nu\tilde{\varepsilon}_{\alpha\alpha} / (1-\nu)$ like in the plane stress problem, what can be done easily in the previous formulation by taking

$$H = 1 - \nu. \tag{14}$$

Furthermore, it can be seen from (11) that the in-plane deformations and the deflection problems are completely uncoupled, hence $u_\alpha(\mathbf{x}) = 0$ provided that only the transversal loading is applied. Then, according to (11), the average stresses satisfy the plane stress conditions and therefore they are referred to as the generalized plane stresses (Barber, 2010; Lurie, 2005).

The case with transversal gradation of material coefficients will be discussed later.

Now in view of (12), (14), (4), the governing equation (8) becomes

$$M_{\alpha\beta,\alpha\beta}(\mathbf{x}) + q(\mathbf{x}) = 0 \quad \text{or} \quad \left\{D(\mathbf{x})\left[Hw_{,\alpha\beta}(\mathbf{x}) + \nu\delta_{\alpha\beta}\nabla^2 w(\mathbf{x})\right]\right\}_{,\alpha\beta} = q(\mathbf{x}) \tag{15}$$

hence,

$$\left[D(\mathbf{x})\nabla^2\nabla^2 w(\mathbf{x}) + 2D_{,\alpha}(\mathbf{x})\nabla^2 w_{,\alpha}(\mathbf{x})\right] + \frac{H}{F}D_{,\alpha\beta}(\mathbf{x})w_{,\alpha\beta}(\mathbf{x}) + \frac{\nu}{F}\left(\nabla^2 D(\mathbf{x})\right)\left(\nabla^2 w(\mathbf{x})\right) = \frac{H}{(1-\nu)F}q(\mathbf{x}) \tag{16}$$

or

$$\nabla^2\left(D(\mathbf{x})\nabla^2 w(\mathbf{x})\right) + \frac{H}{F}\left[D_{,\alpha\beta}(\mathbf{x})w_{,\alpha\beta}(\mathbf{x}) - \left(\nabla^2 D(\mathbf{x})\right)\left(\nabla^2 w(\mathbf{x})\right)\right] = \frac{H}{(1-\nu)F}q(\mathbf{x}), \tag{17}$$

with $F := H + \nu$.

Since the governing equation is the PDE of the 4th order, two relevant fields should be prescribed at each point on the boundary contour $\partial\Omega$ by boundary conditions as described in Eq. (10) and thereafter. Summarizing, we remember that the derived conjugated fields in the semi-integral formulation correspond to the stress state

$$\tilde{\sigma}_{\alpha\beta}(\mathbf{x},x_3) = -\frac{E(\mathbf{x})}{1-\nu^2}\frac{1-\nu}{H}x_3\mu_{\alpha\beta}(\mathbf{x}), \quad \tilde{\sigma}_{\alpha 3}(\mathbf{x},x_3) = 0, \quad \left[\tilde{\sigma}_{33}(\mathbf{x},x_3) = -\frac{E(\mathbf{x})\nu}{1-\nu^2}\frac{1-\nu}{H}x_3 w_{,\alpha\alpha}(\mathbf{x})\right], \tag{18}$$

in the case of transversally loaded plate with invariable Young modulus across the thickness. Although the stresses $\tilde{\sigma}_{33}$ are irrelevant for the considered bending problem and plane stress formulation is applicable in the generalized sense, we have used the symbol H which should be given by (14) in order to obey the plane-stress conditions for relevant stresses.

3. Dimensionless formulation of governing equations

Since the considered theory is linear, it is appropriate to use the dimensionless formulation, because the obtained results are quite universal as regards the geometrical and loading scales. For this purpose, let us use the superstar for dimensionless quantities as

$$x_3^* = \frac{x_3}{h_0}, \quad h^* = \frac{h}{h_0}, \quad w^* = \frac{w}{h_0}, \quad x_\alpha^* = \frac{x_\alpha}{L}, \quad E^* = \frac{E}{E_0}, \quad D^* = \frac{D}{D_0}, \quad \lambda_0^* := q\frac{L^4}{D_0 h_0} \tag{19}$$

where h_0 and L are the characteristic thickness and width of the plate, respectively, D_0 is the bending stiffness of the plate with the constant Young modulus E_0 and thickness h_0,

$$D_0 = \frac{E_0 (h_0)^3}{12(1-\nu^2)} \ , \tag{20}$$

and λ_0^* is the dimensionless transversal loading. Now, the dimensionless counterpart of the governing equation (16) is given as

$$\begin{aligned} &\left[D^*(\mathbf{x}^*)\nabla^2\nabla^2 w^*(\mathbf{x}^*) + 2D^*_{,\alpha}(\mathbf{x}^*)\nabla^2 w^*_{,\alpha}(\mathbf{x}^*)\right] + \frac{H}{F} D^*_{,\alpha\beta}(\mathbf{x}^*) w^*_{,\alpha\beta}(\mathbf{x}^*) + \\ &\qquad + \frac{\nu}{F}\left(\nabla^2 D^*(\mathbf{x}^*)\right)\left(\nabla^2 w^*(\mathbf{x}^*)\right) = \frac{H}{(1-\nu)F}\lambda_0^*(\mathbf{x}^*) \end{aligned} \tag{21}$$

in which the differentiations are performed with respect to the dimensionless coordinates x_α^*. The dimensionless counterparts of the relevant physical fields are expressed as

$$\begin{aligned} &\frac{\partial w^*}{\partial \mathbf{n}}(\mathbf{x}^*) = n_\alpha(\mathbf{x}^*) w^*_{,\alpha}(\mathbf{x}^*) \ , \qquad M^*_{\alpha\beta}(\mathbf{x}^*) := -D^*(\mathbf{x}^*)\frac{1-\nu}{H}\left[H w^*_{,\alpha\beta}(\mathbf{x}^*) + \nu\delta_{\alpha\beta} w^*_{,\gamma\gamma}(\mathbf{x}^*)\right] \\ &M^*(\mathbf{x}^*) = n_\alpha(\mathbf{x}^*) n_\beta(\mathbf{x}^*) M^*_{,\alpha\beta}(\mathbf{x}^*), \qquad T^*(\mathbf{x}^*) = n_\alpha(\mathbf{x}^*) t_\beta(\mathbf{x}^*) M^*_{,\alpha\beta}(\mathbf{x}^*) \\ &V^*(\mathbf{x}^*) := n_\alpha(\mathbf{x}^*) Q_\alpha(\mathbf{x}^*) + t_\alpha(\mathbf{x}^*) T^*_{,\alpha}(\mathbf{x}^*) \\ &Q^*_\alpha(\mathbf{x}^*) := M^*_{\alpha\beta,\beta}(\mathbf{x}^*) = \frac{1-\nu}{H}\left\{H\left[D^*_{,\alpha}(\mathbf{x}^*)\nabla^2 w^*(\mathbf{x}^*) - D^*_{,\beta}(\mathbf{x}^*) w^*_{,\alpha\beta}(\mathbf{x}^*)\right] - F\left(D^*(\mathbf{x}^*)\nabla^2 w^*(\mathbf{x}^*)\right)_{,\alpha}\right\} \end{aligned} \tag{22}$$

Having solved a boundary value problem for the dimensionless governing equation (21), one can obtain the solution of the boundary value problem with the coordinates $(x_\alpha, x_3) = (L x_\alpha^*, h_0 x_3^*)$, and loading $\lambda q(\mathbf{x})$ by using simple rescaling relationships

$$\begin{aligned} &w(\mathbf{x}) = \lambda h_0 w^*(\mathbf{x}^*) \ , \ \frac{\partial w}{\partial \mathbf{n}}(\mathbf{x}) = \frac{\lambda h_0}{L}\frac{\partial w^*}{\partial \mathbf{n}}(\mathbf{x}^*) \ , \ M_{\alpha\beta}(\mathbf{x}) = \frac{\lambda D_0 h_0}{L^2} M^*_{\alpha\beta}(\mathbf{x}^*) \ , \ Q_\alpha(\mathbf{x}) = \frac{\lambda D_0 h_0}{L^3} Q^*_\alpha(\mathbf{x}^*) \ , \\ &V(\mathbf{x}) = \frac{\lambda D_0 h_0}{L^3} V^*(\mathbf{x}^*) \ . \end{aligned} \tag{23}$$

4. Plates with variable bending stiffness

The presented formulation (without any modification) allows spatial variation of the stiffness bending with respect to the in-plane coordinates x_α^*. The Poisson ratio is usually constant, while the Young modulus and the plate thickness can be variable. Thus, according to (13), (19) and (20), we have

$$D^*(\mathbf{x}^*) = E^*(\mathbf{x}^*)\left[h^*(\mathbf{x}^*)\right]^3 \quad . \tag{24}$$

The dependence $h^*(\mathbf{x}^*)$ is the matter of geometrical design, while the continuous variation $E^*(\mathbf{x}^*)$ is the result of mixture of several constituents of multiphase materials with nonuniform microstructure but continuously graded macroproperties usually in a predetermined profile. According to the Voigt's mixture rule, the functional variation of macroproperties corresponds to the variation in the volume fractions of the functionally graded materials (FGM). For instance, using a power-law function to describe the volume fractions of two micro-constituents within the interval $x_1 \in [a, b]$, we have

$$V_{(1)}(x_1^*) = 1 - \left(x_1^* - \frac{a}{L}\right)^p , \quad V_{(2)}(x_1^*) = \left(x_1^* - \frac{a}{L}\right)^p , \quad L := b - a$$

Then, the Young modulus of such a FGM is given as

$$E^*(\mathbf{x}^*) = \frac{1}{E_0}\left[E_{(1)}V_{(1)}(x_1^*) + E_{(2)}V_{(2)}(x_1^*)\right] = 1 + \varepsilon\left(x_1^* - \frac{a}{L}\right)^p , \quad \varepsilon := \frac{E_{(2)}}{E_{(1)}} - 1 , \quad E_0 := E_{(1)} , \tag{25}$$

with $E^*(a/L, x_2^*) = 1 = E_{(1)}/E_0$ and $E^*(b/L, x_2^*) = 1 + \varepsilon = E_{(2)}/E_{(1)} = E_{(2)}/E_0$.

Similar gradation of the Young modulus can be accomplished also in the x_2-direction.

4.1 FGM with gradation across the plate thickness

The gradation of material coefficients across the plate thickness is quite different matter than that along the in-plane directions within the Kirchhoff – Love theory. Note that the assumption $u_\alpha = u_\alpha(\mathbf{x})$ and $w = w(\mathbf{x})$ used in the Kirchhoff hypothesis is in contradiction with allowing $E = E(x_3)$. Ignoring this contradiction, let us try to develop the theory of bending of thin elastic plates with assuming the Kirchhoff hypothesis (1). If the Young modulus in Eq. (3) is given as $E = E(\mathbf{x}, x_3) = E_0 E^V(x_3)\, E^H(\mathbf{x})$,

$$E^V(x_3) = 1 + \zeta\left(\frac{1}{2} + \frac{x_3}{h}\right)^p, \quad \zeta = \frac{E(\mathbf{x}, h/2)}{E(\mathbf{x}, -h/2)} - 1, \text{ then}$$

$$\int_{-h/2}^{h/2} E^V(x_3)dx_3 = \int_{-h/2}^{h/2}\left[1 + \zeta\left(\frac{1}{2} + \frac{x_3}{h}\right)^p\right]dx_3 = h\omega_p, \quad \omega_p := 1 + \frac{\zeta}{p+1}$$

$$\int_{-h/2}^{h/2} x_3 E^V(x_3)dx_3 = \int_{-h/2}^{h/2} x_3\left[1 + \zeta\left(\frac{1}{2} + \frac{x_3}{h}\right)^p\right]dx_3 = h^2\zeta s_p, \quad s_p := \frac{1}{p+2} - \frac{1}{2(p+1)}, \tag{26}$$

$$\int_{-h/2}^{h/2} x_3^2 E^V(x_3)dx_3 = \int_{-h/2}^{h/2} x_3^2\left[1 + \zeta\left(\frac{1}{2} + \frac{x_3}{h}\right)^p\right]dx_3 = \frac{h^3}{12}\beta_p, \quad \beta_p := 1 + \zeta f_p,$$

$$f_p := \frac{12}{p+3} - \frac{12}{p+2} + \frac{3}{p+1},$$

and

$$T_{33}(\mathbf{x}) = \int_{-h/2}^{h/2} \tilde{\sigma}_{33}(\mathbf{x}, x_3)dx_3 = 12D_0\frac{D_H^*(\mathbf{x})\nu}{h_0 h^*}\frac{1-\nu}{H}\left[\frac{\omega_p}{h^* h_0}u_{\alpha,\alpha}(\mathbf{x}) - c\zeta s_p w_{,\alpha\alpha}(\mathbf{x})\right] \tag{27}$$

$$T_{\alpha\beta}(\mathbf{x}) = \int_{-h/2}^{h/2} \tilde{\sigma}_{\alpha\beta}(\mathbf{x}, x_3)dx_3 = 12D_0\frac{D_H^*(\mathbf{x})}{h_0 h^*}\frac{1-\nu}{H}\left[\frac{\omega_p}{h_0 h^*}\tau_{\alpha\beta}(\mathbf{x}) - c\zeta s_p\mu_{\alpha\beta}(\mathbf{x})\right], \tag{28}$$

$$M_{\alpha\beta}(\mathbf{x}) = \int_{-h/2}^{h/2} x_3\tilde{\sigma}_{\alpha\beta}(\mathbf{x}, x_3)dx_3 = D_0 D_H^*(\mathbf{x})\frac{1-\nu}{H}\left[\frac{12}{h^* h_0}c\zeta s_p\tau_{\alpha\beta}(\mathbf{x}) - \beta_p\mu_{\alpha\beta}(\mathbf{x})\right] \tag{29}$$

where $\tau_{\alpha\beta}(\mathbf{x})$, $\mu_{\alpha\beta}(\mathbf{x})$ are given by Eq. (4) and

$$D_H^*(\mathbf{x}) := E^H(\mathbf{x})\left(h^*(\mathbf{x})\right)^3.$$

Eventually, Eqs. (28) and (29) can be rewritten as

$$T_{\alpha\beta}(\mathbf{x}) = 12D_0\frac{1-\nu}{H}\left[\frac{\omega_p}{h_0^2}\left(E^H h^*\right)\tau_{\alpha\beta}(\mathbf{x}) - \frac{c\zeta s_p}{h_0}\left(E^H h^{*2}\right)\mu_{\alpha\beta}(\mathbf{x})\right] \tag{28'}$$

$$M_{\alpha\beta}(\mathbf{x}) = -D_0 D_V^* D_H^*(\mathbf{x})\frac{1-\nu}{H}\mu_{\alpha\beta}(\mathbf{x}) + \frac{c\zeta s_p}{\omega_p}h_0 h^* T_{\alpha\beta}(\mathbf{x}), \tag{29'}$$

with $D_V^* := \beta_p - 12\dfrac{(c\zeta s_p)^2}{\omega_p}$.

Note that the factor c is equal to one, but we introduce it in order to distinguish the interaction terms.

Now, according to Eqs. (27) – (29), the governing equations (7) and (8) are not uncoupled any more for the in-plane deformations and the bending deformations as long as $\zeta \neq 0$. Furthermore, $T_{33}(\mathbf{x}) \approx T_{\alpha\beta}(\mathbf{x})$, hence the plane stress conditions are not satisfied either in the generalized sense. Really, the role of $T_{33}(\mathbf{x})$ can be discarded because of $\tilde{\varepsilon}_{33} = 0$. Then, in view of (27), we have

$$u_{\alpha,\alpha}(\mathbf{x}) = \frac{c\zeta s_p}{\omega_p} h w_{,\alpha\alpha}(\mathbf{x}) \quad \text{and} \quad \tilde{\varepsilon}_{\alpha\alpha} = u_{\alpha,\alpha} - x_3 w_{,\alpha\alpha} = \left(\frac{c\zeta s_p}{\omega_p} h - x_3 \right) w_{,\alpha\alpha}(\mathbf{x}) \ .$$

Apparently, the relationship $\tilde{\varepsilon}_{33} = -\nu \tilde{\varepsilon}_{\alpha\alpha} / (1-\nu)$ cannot be satisfied either in the mid-plane of the plate as long as there is a gradation of the Young modulus ($\zeta \neq 0$). Thus, the application of the plane stress formulation is questionable for FGM plates with transversal gradation of the Young modulus.

The system of coupled governing equations is given by the system of partial differential equations

$$\frac{\omega_p}{h_0^2}\left[\left(E^H h^*\right)_{,\beta} \tau_{\alpha\beta}(\mathbf{x}) + \left(E^H h^*\right) \tau_{\alpha\beta,\beta}(\mathbf{x}) \right] - \frac{c\zeta s_p}{h_0}\left[\left(E^H h^{*2}\right)_{,\beta} \mu_{\alpha\beta}(\mathbf{x}) + \left(E^H h^{*2}\right) \mu_{\alpha\beta,\beta}(\mathbf{x}) \right] = 0 \quad (30)$$

$$\begin{aligned} D_H^*(\mathbf{x})\mu_{\alpha\beta,\beta\alpha}(\mathbf{x}) + 2D_{H,\beta}^*(\mathbf{x})\mu_{\alpha\beta,\alpha}(\mathbf{x}) + D_{H,\alpha\beta}^*(\mathbf{x})\mu_{\alpha\beta}(\mathbf{x}) - \frac{H}{1-\nu}\frac{h_0 c\zeta s_p}{\omega_p D_0 D_V^*} h_{,\alpha\beta}^*(\mathbf{x}) T_{\alpha\beta}(\mathbf{x}) = \\ = \frac{H}{1-\nu}\frac{q(\mathbf{x})}{D_0 D_V^*} \ . \end{aligned} \quad (31)$$

Certain simplification of the system of governing equations (30) and (31) can be achieved when the thickness of the plate is constant (then, $D_H^*(\mathbf{x}) = E^H(\mathbf{x})$) or the Young modulus is not graded horizontally (then, $D_H^*(\mathbf{x}) = \left(h^*(\mathbf{x})\right)^3$). Obviously, the in-plane deformations are decoupled, if there is no gradation across the plate thickness. On the other hand, if the Young modulus is variable only across the constant plate thickness, i.e. when $E^H(\mathbf{x}) = 1$ and $h^* = 1$, this system of PDE is significantly simplified

$$\omega_p \tau_{\alpha\beta,\beta}(\mathbf{x}) - c\zeta s_p h_0 h^* \mu_{\alpha\beta,\beta}(\mathbf{x}) = 0 \quad (32)$$

$$D_0 D_V^* \mu_{\alpha\beta,\beta\alpha}(\mathbf{x}) = \frac{H}{1-\nu} q(\mathbf{x}) \ . \quad (33)$$

Although this system is coupled, the bending problem can be solved separately and subsequently one can solve the in-plane problem.

Alternatively, in view of (4) Eq. (33) can be rewritten as

$$D_0 D_V^* \nabla^2 \nabla^2 w(\mathbf{x}) = \frac{H}{(1-\nu)F} q(\mathbf{x}) \ , \quad (34)$$

Remark 1

Note that if the plane stress formulation is applied, $H = 1 - \nu$ and $F = 1$, hence the loading factor

$$\frac{H}{(1-\nu)F}=1 .$$

On the other hand, if the plane stress formulation was not applied, we would have

$$\frac{H}{(1-\nu)F}=\frac{1-2\nu}{(1-\nu)^2},$$

since in that case $H = 1 - 2\nu$ and $F = 1 - \nu$. The factor $(1 - \nu)^2 / (1 - 2\nu)$ is equal to 1.225 for $\nu = 0.3$. The correct value for H could be easily justified by comparison of computational results for deflections with measured values, since the difference between the computed values corresponding to $H = 1 - \nu$ and $H = 1 - 2\nu$ is around 20 % (see also Part II of this paper for illustration).

Remark 2

In order to assess the effect of the coupling of the deflection and in-plane deformations on the bending stiffness and finally on the solution of the bending problem in the FGM plates with $E = E(x_3)$, one should solve the problem twice with taking the interaction parameter $c = 1$ as well as $c = 0$, though the solution for $c = 0$ corresponds to an incorrect treatment of the gradation $E = E(x_3)$, leading to incorrect value $D_V^* = \beta_p$ and it is considered just for comparison.

Remark 3

Another interesting point is the independence of the bending of the thin elastic plate on the orientation of the gradation of the Young modulus across the plate thickness. Really, let us consider the gradation

$$E = E(x_3) = E_0 E^V(x_3),\ E^V(x_3) = 1+\zeta\left(\frac{1}{2}-\frac{x_3}{h}\right)^p,\ E(-h/2) = (1+\zeta)E(h/2) = (1+\zeta)E_0 \tag{35}$$

in contrast to the previous gradation, where

$$E = E(x_3) = E_0 E^V(x_3),\ E^V(x_3) = 1+\zeta\left(\frac{1}{2}+\frac{x_3}{h}\right)^p \text{ and } E(h/2) = (1+\zeta)E(-h/2) = (1+\zeta)E_0 . \tag{36}$$

It can be easily verified that all equations (26) – (34) remain valid, with s_p being replaced by $-s_p$. Such a replacement, however, does not change D_V^*. In other words, it does not matter for bending of the FGM plate if it is tougher in the tension zone or in the compression zone.

Thus, the bending problem in FGM plates within the Kirchhoff – Love theory can be solved separately from the in-plane deformation problem only if the thickness of the plate is invariable and the Young modulus is dependent either on the in-plane coordinates, $E = E(\mathbf{x})$, or on the transversal coordinate, $E = E(x_3)$. Then, the problem with $E = E(x_3)$ becomes simpler than the problem with $E = E(\mathbf{x})$, because it is partially equivalent to the bending problem in homogeneous plate with the same distribution of transversal loading, but the correct value of the bending stiffness must be considered. This correct value (34) incorporates not only the transversal gradation of the Young modulus but also the coupling of the deflections with the in-plane deformations.

4.2 Dimensionless formulation and decreasing the order of the derivatives

Making use the definitions of the dimensionless deflections, displacements and coordinates, we can rewrite the averaged stress tensor and stress couples as

$$T_{\alpha\beta}(\mathbf{x}) = \frac{D_0}{L^2} T^*_{\alpha\beta}(\mathbf{x}^*), \qquad T^*_{\alpha\beta} = 12\frac{1-\nu}{H}\left\{\frac{\omega_p L}{h_0}\left(E^H h^*\right)\tau^*_{\alpha\beta} - c\zeta s_p\left(E^H h^{*2}\right)\left[H w^*_{,\alpha\beta} + \nu\delta_{\alpha\beta}\nabla^2 w^*\right]\right\},$$

$$M_{\alpha\beta}(\mathbf{x}) = \frac{h_0 D_0}{L^2} M^*_{\alpha\beta}(\mathbf{x}^*), \qquad M^*_{\alpha\beta} = -\frac{1-\nu}{H} D^*\left[H w^*_{,\alpha\beta} + \nu\delta_{\alpha\beta}\nabla^2 w^*\right] + \frac{c\zeta s_p}{\omega_n} h^* T^*_{\alpha\beta}$$

with $\tau^*_{\alpha\beta} := \frac{H}{2}\left(u^*_{\alpha,\beta} + u^*_{\beta,\alpha}\right) + \nu\delta_{\alpha\beta}u^*_{\gamma,\gamma} = \frac{L}{h_0}\tau_{\alpha\beta}$, $D^*(\mathbf{x}^*) := D^*_V D^*_H(\mathbf{x}^*)$, $D(\mathbf{x}) = D_0 D^*(\mathbf{x}^*)$

In order to eliminate the expected 4th and 3rd order derivatives of deflections, let us introduce a new field variable as

$$m^*(\mathbf{x}^*) := -D^*(\mathbf{x}^*)\nabla^2 w^*(\mathbf{x}^*) \quad \text{or} \quad m(\mathbf{x}) = -D(\mathbf{x})\nabla^2 w(\mathbf{x}) = \frac{h_0 D_0}{L^2} m^*(\mathbf{x}^*) \ . \tag{37}$$

Then,
$$\begin{aligned} T^*_{\alpha\beta} &= 12\frac{1-\nu}{H}\left\{\frac{\omega_p L}{h_0}\left(E_H h^*\right)\tau^*_{\alpha\beta} - c\zeta s_p\left[H\left(E_H h^{*2}\right)w^*_{,\alpha\beta} - \delta_{\alpha\beta}\frac{\nu}{D^*_V}\frac{m^*}{h^*}\right]\right\} = \\ &= 12\frac{1-\nu}{H}\frac{\omega_p L}{h_0 D^*_V}\left\{\frac{D^*}{h^{*2}}\tau^*_{\alpha\beta} - \frac{\gamma_p}{h^*}\left[H D^* w^*_{,\alpha\beta} - \nu\delta_{\alpha\beta}m^*\right]\right\}, \qquad \gamma_p := \frac{c\zeta s_p h_0}{\omega_p L} \end{aligned} \tag{38}$$

$$\begin{aligned} M^*_{\alpha\beta} &= \frac{1-\nu}{H}\left[-D^* H w^*_{,\alpha\beta} + \nu\delta_{\alpha\beta}m^*\right] + \frac{c\zeta s_p}{\omega_p} h^* T^*_{\alpha\beta} = \\ &= \frac{1-\nu}{H}\frac{\beta_p}{D^*_V}\left[-D^* H w^*_{,\alpha\beta} + \nu\delta_{\alpha\beta}m^*\right] + \frac{1-\nu}{H}\alpha_p\frac{D^*}{h^* D^*_V}\tau^*_{\alpha\beta} \ , \qquad \alpha_p := 12\frac{c\zeta s_p L}{h_0} \end{aligned} \tag{39}$$

and the governing equations $T^*_{\alpha\beta,\beta} = 0$, $M^*_{\alpha\beta,\alpha\beta} = -\lambda^*_0 = -\frac{L^4}{h_0 D_0} q$ become

$$\begin{aligned} &\frac{\omega_p L}{h_0}\left[\left(E_H h^*\right)_{,\beta}\tau^*_{\alpha\beta} + \left(E_H h^*\right)\tau^*_{\alpha\beta,\beta}\right] + \\ &+ c\zeta s_p\left[\frac{F}{D^*_V h^*}m^*_{,\alpha} - \frac{m^*}{D^*_V h^*}\left(\nu\frac{h^*_{,\alpha}}{h^*} + H\frac{D^*_{,\alpha}}{D^*}\right) - H\left(E_H h^{*2}\right)_{,\beta} w^*_{,\alpha\beta}\right] = 0 \end{aligned} \tag{40}$$

$$D^*\nabla^2 w^* + m^* = 0 \tag{41}$$

$$\nabla^2 m^* + \left[\alpha_p \gamma_p \frac{\nu}{F D_V^*} \frac{\nabla^2 h^*}{h^*} - \frac{H}{F} \frac{\nabla^2 D^*}{D^*}\right] m^* - \frac{H}{F}\left[D^*_{,\alpha\beta} + \alpha_p \gamma_p \frac{D^*}{D_V^*} \frac{h^*_{,\alpha\beta}}{h^*}\right] w^*_{,\alpha\beta} + \\ + \frac{\alpha_p}{F} \frac{D^*}{D_V^*} \frac{h^*_{,\alpha\beta}}{h^{*2}} \tau^*_{\alpha\beta} = -\frac{H}{(1-\nu)F} \lambda_0^* . \tag{42}$$

Recall that the order of the derivatives in the governing PDE does not exceed the second order. Finally, we bring the expressions for the physical quantities occurring in the boundary conditions in terms of the field variables $w^*(\mathbf{x}^*)$, $m^*(\mathbf{x}^*)$, $u_\alpha^*(\mathbf{x})$ and their derivatives. At an arbitrary boundary point, we may write the following expressions:

for the dimensionless tractions

$$T_\alpha^* := n_\beta T^*_{\alpha\beta} = 12 \frac{1-\nu}{H} \frac{\omega_p L}{h_0 D_V^*} \left\{ \frac{D^*}{h^{*2}} \tau^*_{\alpha\beta} n_\beta - \frac{\gamma_p}{h^*} \left[H D^* w^*_{,\alpha\beta} n_\beta - \nu n_\alpha m^* \right] \right\}, \tag{43}$$

for the dimensionless bending moment

$$M^* = n_\alpha n_\beta M^*_{\alpha\beta} = \frac{1-\nu}{H} \frac{\beta_p}{D_V^*} \left[-D^* H n_\alpha n_\beta w^*_{,\alpha\beta} + \nu m^* \right] + \frac{1-\nu}{H} \alpha_p \frac{D^*}{h^* D_V^*} n_\alpha n_\beta \tau^*_{\alpha\beta} \ , \tag{44}$$

or the twisting moment

$$T^* := n_\alpha t_\beta M^*_{\alpha\beta} = -(1-\nu)\beta_p \frac{D^*}{D_V^*} n_\alpha t_\beta w^*_{,\alpha\beta} + \frac{1-\nu}{H} \alpha_p \frac{D^*}{h^* D_V^*} n_\alpha t_\beta \tau^*_{\alpha\beta} = \\ = -(1-\nu)\beta_p \frac{D^*}{D_V^*} \frac{\partial}{\partial \mathbf{n}} \frac{\partial w^*}{\partial \mathbf{t}} + (1-\nu)\alpha_p \frac{D^*}{2h^* D_V^*} \left(n_\alpha \frac{\partial u_\alpha^*}{\partial \mathbf{t}} + t_\beta \frac{\partial u_\beta^*}{\partial \mathbf{n}} \right) \tag{45}$$

and finally for the generalized shear force

$$V^* = n_\alpha M^*_{\alpha\beta,\beta} + \frac{\partial T^*}{\partial \mathbf{t}} \ ,$$

with

$$n_\alpha M^*_{\alpha\beta,\beta} = \frac{1-\nu}{H} \left[F n_\alpha m^*_{,\alpha} - H \frac{D^*_{,\alpha}}{D^*} n_\alpha m^* - H n_\alpha D^*_{,\beta} w^*_{,\alpha\beta} \right] + \\ + \alpha_p \gamma_p \frac{1-\nu}{H D_V^*} \frac{h^*_{,\beta}}{h^*} n_\alpha \left[-H D^* w^*_{,\alpha\beta} + \nu \delta_{\alpha\beta} m^* \right] + \frac{1-\nu}{H} \alpha_p \frac{D^*}{D_V^*} \frac{h^*_{,\beta}}{(h^*)^2} n_\alpha \tau^*_{\alpha\beta} \ . \tag{46}$$

Note that the in-plane tractions $n_\beta T^*_{\alpha\beta}$ are assumed to be vanishing on the simply supported (SSE) and/or free edge (FE). Then, the bending moment as well as the generalized shear force on such edges can be expressed in terms of the field variables $w^*(\mathbf{x}^*)$, $m^*(\mathbf{x}^*)$ and their derivatives alone. Thus, in view of (39) we obtain the following simplified expressions on the SSE and FE

$$M^* = n_\alpha n_\beta M^*_{\alpha\beta} = \frac{1-\nu}{H}\left[-D^* H n_\alpha n_\beta w^*_{,\alpha\beta} + \nu m^*\right] \tag{47}$$

$$T^* = -(1-\nu)D^* n_\alpha t_\beta w^*_{,\alpha\beta} = -(1-\nu)D^* \frac{\partial}{\partial \mathbf{n}}\frac{\partial w^*}{\partial \mathbf{t}} ,$$

$$M^*_{\alpha\beta,\beta} = \frac{1-\nu}{H}\left[-D^*_{,\beta} H w^*_{,\alpha\beta} + F m^*_{,\alpha} - H\frac{D^*_{,\alpha}}{D^*} m^*\right] + \frac{c\zeta s_p}{\omega_p} h^*_{,\beta} T^*_{\alpha\beta} , \tag{48}$$

hence
$$n_\alpha M^*_{\alpha\beta,\beta} = (1-\nu)\left[-D^*_{,\beta} n_\alpha w^*_{,\alpha\beta} + \frac{F}{H}\frac{\partial m^*}{\partial \mathbf{n}} - \frac{1}{D^*}\frac{\partial D^*}{\partial \mathbf{n}} m^*\right] = $$
$$= (1-\nu)\left[-\frac{\partial D^*}{\partial \mathbf{n}}\frac{\partial}{\partial \mathbf{n}}\frac{\partial w^*}{\partial \mathbf{n}} - \frac{\partial D^*}{\partial \mathbf{t}}\frac{\partial}{\partial \mathbf{n}}\frac{\partial w^*}{\partial \mathbf{t}} + \frac{F}{H}\frac{\partial m^*}{\partial \mathbf{n}} - \frac{1}{D^*}\frac{\partial D^*}{\partial \mathbf{n}} m^*\right], \tag{49}$$

$$V^* = \frac{(1-\nu)F}{H}\frac{\partial m^*}{\partial \mathbf{n}} - (1-\nu)\left[\frac{1}{D^*}\frac{\partial D^*}{\partial \mathbf{n}} m^* + \frac{\partial D^*}{\partial \mathbf{n}}\frac{\partial}{\partial \mathbf{n}}\frac{\partial w^*}{\partial \mathbf{n}} + 2\frac{\partial D^*}{\partial \mathbf{t}}\frac{\partial}{\partial \mathbf{n}}\frac{\partial w^*}{\partial \mathbf{t}}\right] - (1-\nu)D^*\frac{\partial}{\partial \mathbf{t}}\frac{\partial}{\partial \mathbf{n}}\frac{\partial w^*}{\partial \mathbf{t}} . \tag{50}$$

Remember the third order derivative of deflections in the expressions for the generalized shear force, but it should be said that it disappears in problems exhibiting the translational symmetry along the free edge, when $\partial w^* / \partial \mathbf{t} = 0$.

Remark 4

In the case of thin elastic plate with constant thickness and the Young modulus graded across the plate thickness one can formulate a "bending boundary value problem" which can be solved separately from the in-plane deformation problem, because the governing equations for the fields w^* and m^* do not involve the field u^*_α and the relevant boundary densities are expressed on particular portions of the boundary edge in terms of the fields w^* and m^* as well. Nevertheless, the complete solution of this coupled problem is required even if we were not interested in the in-plane deformations. These are needed also for getting the bending moment and/or the generalized shear force on clamped edges.

5. Angularly symmetric bending of circular plates

In what follows, we shall consider circular thin elastic plates subjected to transversal loading and boundary conditions exhibiting angular symmetry. Both the transversal and in-plane gradation of the Young modulus as well as variable plate thickness are allowed. Because of the symmetry, the considered problem is simplified when polar coordinates (r, φ) are employed instead of Cartesian coordinates $(x_1, x_2) = (r\cos\varphi, r\sin\varphi)$ and $\partial(.)/\partial\varphi \equiv 0$. Then, for arbitrary differentiable function $f(r)$ we have

$$f_{,\alpha}(r) = r_{,\alpha}\frac{\partial f(r)}{\partial r}\ ,\ f_{,\alpha\beta}(r) = r_{,\alpha\beta}\frac{\partial f(r)}{\partial r} + r_{,\alpha}r_{,\beta}\frac{\partial^2 f(r)}{\partial r^2},\quad r_{,\alpha\beta} = \frac{1}{r}\left(\delta_{\alpha\beta} - r_{,\alpha}r_{,\beta}\right)$$
$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right). \tag{51}$$

Since the boundary edge Γ is a circle, the outer unit normal vector on Γ is $n_\alpha = \pm r_{,\alpha}$, where the lower sign is valid on the inner boundary edge in the case of the circular plate with central circular hole. The unit tangent vector on the boundary edge is $t_\alpha = \varepsilon_{3\beta\alpha}n_\beta = \pm\varepsilon_{3\beta\alpha}r_{,\beta}$. Now, in view of (51) we have on the boundary edges

$$f_{,\alpha}(r) = r_{,\alpha}\frac{\partial f(r)}{\partial r} = \pm n_\alpha\frac{\partial f(r)}{\partial r},$$
$$f_{,\alpha\beta}(r) = t_\alpha t_\beta\frac{1}{r}\frac{\partial f(r)}{\partial r} + n_\alpha n_\beta\frac{\partial^2 f(r)}{\partial r^2} = n_\alpha n_\beta\nabla^2 f(r) + \left(t_\alpha t_\beta - n_\alpha n_\beta\right)\frac{1}{r}\frac{\partial f(r)}{\partial r}. \tag{52}$$

In order to rewrite the governing equations (40) – (42) and the expressions for the relevant boundary quantities into polar coordinate system, we give some helpful formulae

$$u^*_{\alpha,\beta} = \left(r_{,\alpha}u^*_r\right)_{,\beta} = r_{,\alpha\beta}u^*_r + r_{,\alpha}r_{,\beta}\partial_r u^*_r = u^*_{\beta,\alpha}\,,\qquad u^*_{\gamma,\gamma} = \left(\frac{1}{r} + \partial_r\right)u^*_r$$
$$u^*_{\beta,\beta\alpha} = r_{,\alpha}\partial_r u^*_{\beta,\beta} = r_{,\alpha}\left(\nabla^2 - \frac{1}{r^2}\right)u^*_r,\qquad \nabla^2 u_\alpha = \nabla^2\left(r_{,\alpha}u^*_r\right) = r_{,\alpha}\left(\nabla^2 - \frac{1}{r^2}\right)u^*_r$$
$$\tau^*_{\alpha\beta} = H\left(r_{,\alpha\beta} + r_{,\alpha}r_{,\beta}\partial_r\right)u^*_r + \nu\delta_{\alpha\beta}\left(\frac{1}{r} + \partial_r\right)u^*_r,\qquad r_{,\beta}\tau^*_{\alpha\beta} = r_{,\alpha}\left(F\partial_r + \frac{\nu}{r}\right)u^*_r$$
$$\tau^*_{\alpha\beta,\beta} = \frac{H}{2}\nabla^2 u^*_\alpha + \frac{F+\nu}{2}u^*_{\beta,\beta\alpha} = F\nabla^2 u^*_\alpha = F\,r_{,\alpha}\left(\nabla^2 - \frac{1}{r^2}\right)u^*_r \tag{53}$$
$$w^*_{,\alpha} = \left(r_{,\alpha}w^*\right)_{,\beta} = r_{,\alpha}\partial_r w^*,\quad w^*_{,\alpha\beta} = \left(r_{,\alpha\beta}\partial_r + r_{,\alpha}r_{,\beta}\partial_r^2\right)w^*,\quad \nabla^2 w^* = w^*_{,\gamma\gamma} = \left(\frac{1}{r}\partial_r + \partial_r^2\right)w^*$$
$$r_{,\beta}w^*_{,\alpha\beta} = r_{,\alpha}\partial_r^2 w^* = -r_{,\alpha}\left(\frac{m^*}{D^*} + \frac{1}{r}\partial_r w^*\right),\quad r_{,\alpha\beta}w^*_{,\alpha\beta} = r_{,\alpha\beta}\left(r_{,\alpha\beta}\partial_r + r_{,\alpha}r_{,\beta}\partial_r^2\right)w^* = \frac{1}{r^2}\partial_r w^*$$
$$r_{,\alpha}r_{,\beta}w^*_{,\alpha\beta} = r_{,\alpha}r_{,\beta}\left(r_{,\alpha\beta}\partial_r + r_{,\alpha}r_{,\beta}\partial_r^2\right)w^* = \partial_r^2 w^* = -\left(\frac{m^*}{D^*} + \frac{1}{r}\partial_r w^*\right).$$

Now, the governing equations for axially symmetric problems become

$$\frac{\omega_p L}{h_0}\left[\left(E_H h^*\right)F\partial_r^2 u_r^* + F\left(\frac{\partial(E_H h^*)}{\partial r}+\frac{E_H h^*}{r}\right)\partial_r u_r^* + \left(\frac{\nu}{r}\frac{\partial(E_H h^*)}{\partial r} - F\frac{E_H h^*}{r^2}\right)u_r^*\right]+$$
$$+c\zeta s_p\left\{\frac{H}{r}\frac{\partial\left(E_H h^{*2}\right)}{\partial r}\partial_r w^* + \frac{F}{D_V^* h^*}\partial_r m^* + \left[\frac{H}{D^*}\frac{\partial\left(E_H h^{*2}\right)}{\partial r} - \frac{1}{D_V^* h^*}\left(\nu\frac{\partial_r h^*}{h^*} + H\frac{\partial_r D^*}{D^*}\right)\right]\right. \tag{54}$$

$$D^*\nabla^2 w^* + m^* = 0 \tag{55}$$

$$\nabla^2 m^* + \left[\frac{\alpha_p \gamma_p}{F D_V^* h^*}\left(\nu\nabla^2 h^* + H\partial_r^2 h^*\right) - \frac{H}{F D^* r}\partial_r D^*\right]m^* -$$
$$-\frac{H}{Fr}\left[\alpha_p \gamma_p \frac{D^*}{D_V^* h^*}\left(\frac{1}{r}\partial_r h^* - \partial_r^2 h^*\right) + \frac{1}{r}\partial_r D^* - \partial_r^2 D^*\right]\partial_r w^* +$$
$$+\frac{\alpha_p}{F}\frac{D^*}{D_V^* h^{*2}}\left\{\left[\frac{F}{r^2}\partial_r h^* + \frac{\nu}{r}\partial_r^2 h^*\right]u_r^* + \left[\frac{\nu}{r}\partial_r h^* + F\partial_r^2 h^*\right]\partial_r u_r^*\right\} = -\frac{H}{(1-\nu)F}\lambda_0^* . \tag{56}$$

The system of governing equations (54) – (56) is significantly simplified if the Young modulus is not graded across the plate thickness, i.e. when $\zeta = 0$. Then, the deflections and radial displacements are decoupled with the governing equations for the bending being given as

$$D^*\nabla^2 w^* + m^* = 0\,, \qquad D^*(\mathbf{x}^*) = D_H^*(\mathbf{x}^*) = E_H(\mathbf{x}^*)\left(h^*(\mathbf{x}^*)\right)^3 \tag{57}$$

$$\nabla^2 m^* - \frac{H}{F D^* r}\left(\partial_r D^*\right)m^* - \frac{H}{Fr}\left(\frac{1}{r}\partial_r D^* - \partial_r^2 D^*\right)\partial_r w^* = -\frac{H}{(1-\nu)F}\lambda_0^* \tag{58}$$

Recall that plane stress formulation ($H = 1 - \nu$, $F = 1$) is applicable in this case.

Another significant simplification of the system of governing equations occurs when $E_H = 1$ and $h^* = 1$. In that case the governing equations for axially symmetric problems become

$$\frac{\omega_p L}{h_0}\left(\partial_r^2 u_r^* + \frac{1}{r}\partial_r u_r^* - \frac{1}{r^2}u_r^*\right) + \frac{c\zeta s_p}{D_V^*}\partial_r m^* = 0 \tag{59}$$

$$D^*\nabla^2 w^* + m^* = 0 \tag{60}$$

$$\nabla^2 m^* = -\frac{H}{(1-\nu)F}\lambda_0^* . \tag{61}$$

For the sake of completeness, it is necessary to present the expressions for relevant boundary quantities. Firstly, we start with the expressions applicable at an arbitrary point on the boundary edges. The in-plane tractions are given as

$$T^*_{\alpha\beta}n_\beta = \pm r_{,\alpha} 12\frac{1-\nu}{H}\frac{\omega_p L}{h_0 D^*_V}\left\{\frac{D^*}{h^{*2}}\left(F\frac{\partial}{\partial r}+\frac{\nu}{r}\right)u^*_r + \frac{\gamma_p}{h^*}\left(Fm^* + \frac{HD^*}{r}\frac{\partial w^*}{\partial r}\right)\right\}, \tag{62}$$

the dimensionless bending moment as

$$M^* = n_\alpha n_\beta M^*_{\alpha\beta} = \frac{1-\nu}{H}\frac{\beta_p}{D^*_V}\left(Fm^* + \frac{D^* H}{r}\partial_r w^*\right) + \frac{1-\nu}{H}\alpha_p \frac{D^*}{h^* D^*_V}\left(F\partial_r + \frac{\nu}{r}\right)u^*_r \ , \tag{63}$$

the twisting moment as

$$T^* := n_\alpha t_\beta M^*_{\alpha\beta} = -(1-\nu)\beta_p \frac{D^*}{D^*_V} n_\alpha t_\beta w^*_{,\alpha\beta} = 0 \, , \tag{64}$$

and finally the generalized shear force as

$$\begin{aligned} V^* = n_\alpha M^*_{\alpha\beta,\beta} + \frac{\partial T^*}{\partial \mathbf{t}} = n_\alpha M^*_{\alpha\beta,\beta} = \pm(1-\nu)\left(\frac{F}{H}\partial_r m^* - \frac{\partial_r D^*}{r}\partial_r w^*\right) + \\ \pm\frac{1-\nu}{H}\frac{\alpha_p \gamma_p}{D^*_V}\frac{\partial_r h^*}{h^*}\left(Fm^* + \frac{HD^*}{r}\partial_r w^*\right) \pm \frac{1-\nu}{H}\frac{\alpha_p D^*}{D^*_V}\frac{\partial_r h^*}{(h^*)^2}\left(F\partial_r + \frac{\nu}{r}\right)u^*_r \, . \end{aligned} \tag{65}$$

Eventually, on the simply supported and/or free edges, the in-plane tractions vanish [$T^*_{\alpha\beta}n_\beta = 0$ in Eq. (62)] and the expressions for the bending moment and the generalized shear force can be written as

$$M^* = (1-\nu)\left[-D^* n_\alpha n_\beta w^*_{,\alpha\beta} + \frac{\nu}{H}m^*\right] = (1-\nu)\left[-D^*\frac{\partial^2 w^*}{\partial r^2} + \frac{\nu}{H}m^*\right] = (1-\nu)\left[\frac{D^*}{r}\frac{\partial w^*}{\partial r} + \frac{F}{H}m^*\right], \tag{66}$$

$$V^* = n_\alpha M^*_{\alpha\beta,\beta} = \pm(1-\nu)\left(\frac{F}{H}\partial_r m^* - \frac{\partial_r D^*}{D^*}m^* - (\partial_r D^*)\partial_r^2 w^*\right) = \pm(1-\nu)\left(\frac{F}{H}\partial_r m^* + \frac{\partial_r D^*}{r}\partial_r w^*\right) . \tag{67}$$

Note that the relevant boundary quantities do not involve higher than the first order derivatives of the used field variables.

6. Exact solutions for circular plate

6.1 Plate with variable bending stiffness

Let us consider a circular plate with a central circular hole under a homogeneous transversal loading $\lambda_0^*(r^*) = 1$. The outer and inner radii are r_1^* and r_0^* , respectively, and the bending stiffness is assumed to be power-law graded $D^*(r^*) = \left(r/r_0\right)^n = \left(r^*/r_0^*\right)^n$. Bearing in mind the plane stress conditions and taking $\lambda_0^* = 1$, one can rewrite Eq. (58) as

$$\nabla^2 m^* + \frac{1-\nu}{r}\frac{\partial}{\partial r}\left(\frac{\partial D^*}{\partial r}\frac{\partial w^*}{\partial r}\right) = -1$$

and finally substitute (57) to obtain the ordinary differential equation (ODE)

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial(D^*\nabla^2 w^*)}{\partial r^*}\right) - \frac{1-\nu}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial D^*}{\partial r^*}\frac{\partial w^*}{\partial r^*}\right) = 1 \ . \tag{68}$$

hence, after integrating, one gets the ODE for $\varphi(r^*) := \partial w^*(r^*) / \partial r^*$

$$\hat{A}\varphi = r_0^{*n}\left(\frac{r^{*(1-n)}}{2} + \frac{C}{r^{*(1+n)}}\right), \quad \hat{A} = \frac{\partial^2}{\partial r^{*2}} + (1+n)\frac{1}{r^*}\frac{\partial}{\partial r^*} + (n\nu - 1)\frac{1}{r^{*2}}, \tag{69}$$

where C is an integration constant. Now, one can easily find the particular solution as well as the homogeneous solution for the ODE (69)

$$\frac{\partial w^*}{\partial r^*}(r^*) = Qr^{*(3-n)} + B_1 r^{*\beta_1} + B_2 r^{*\beta_2} + B_3 r^{*(1-n)}, \tag{70}$$

where B_1, B_2, B_3 are the integration constants determined by the prescribed boundary conditions, while Q and β_1, β_2 are given as

$$Q = \frac{(r_0^*)^n}{2[8+n(\nu-3)]}, \quad \beta_1 = -\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + 1 - n\nu}, \quad \beta_2 = -\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + 1 - n\nu} \ . \tag{71}$$

For further integration of Eq. (70), it is necessary to distinguish three particular cases with respect to values of the gradation parameter n.

(i) $\underline{n \neq 2 \wedge n \neq 4:}$

Recall that in this case $\beta_{1(2)} \neq -1$ as long as $n \neq 0$, since $\nu \neq 1$, because $\nu \in [-1,1/2]$. Then, starting from (70), we obtain

$$w^*(r^*) = \frac{Q}{4-n} r^{*(4-n)} + \frac{B_1}{\beta_1+1} r^{*(\beta_1+1)} + \frac{B_2}{\beta_2+1} r^{*(\beta_2+1)} + \frac{B_3}{2-n} r^{*(2-n)} + B_4$$

$$\frac{\partial^2}{\partial r^{*2}} w^*(r^*) = (3-n)Qr^{*(2-n)} + \beta_1 B_1 r^{*(\beta_1-1)} + \beta_2 B_2 r^{*(\beta_2-1)} + (1-n)B_3 / r^{*n} \tag{72}$$

$$m^*(r^*) = -\frac{1}{(r_0^*)^n}\left[(4-n)Qr^{*2} + (1+\beta_1)B_1 r^{*(n+\beta_1-1)} + (1+\beta_2)B_2 r^{*(n+\beta_2-1)} + (2-n)B_3\right]$$

$$\frac{\partial}{\partial r^*} m^*(r^*) = -\frac{1}{(r_0^*)^n}\left[2(4-n)Qr^* + (1+\beta_1)(n+\beta_1-1)B_1 r^{*(n+\beta_1-2)} + (1+\beta_2)(n+\beta_2-1)B_2 r^{*(n+\beta_2-2)}\right]$$

$$M^*(r^*) = -\frac{1}{(r_0^*)^n}\left[(3+\nu-n)Qr^{*2} + (\beta_1+\nu)B_1 r^{*(n+\beta_1-1)} + (\beta_2+\nu)B_2 r^{*(n+\beta_2-1)} + (1+\nu-n)B_3\right] \qquad (72)$$

$$V^*(r^*) = \mp\frac{1}{r_0^{*n}}\left\{\left[8-(3-\nu)n\right]Qr^* + \tilde{\beta}_1 B_1 r^{*(n+\beta_1-2)} + \tilde{\beta}_2 B_2 r^{*(n+\beta_2-2)} - n(1-\nu)\frac{B_3}{r^*}\right\}$$

with $\tilde{\beta}_i := (1+\beta_i)(n+\beta_i-1) - n(1-\nu)$.

Since the plate without any support (with both edges to be free) is meaningless, the deflection is prescribed anyway at least on one edge and the integration constant B_4 can be found from the prescribed deflection after having known the other three integration constant B_1, B_2, B_3. The rest three boundary conditions give us the system of three algebraic equations $A_{ij}B_k = P_i$, $(i,k = 1,2,3)$ used for determination of B_1, B_2, B_3. In what follows, we shall specify A_{ik} and P_i in particular boundary value problems:

Clamped edge r_0 – Clamped edge r_1 $[\, w^*(r_0^*) = 0,\ \partial_r w^*(r_0^*) = 0,\ w^*(r_1^*) = 0,\ \partial_r w^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & r_0^{*(1-n)} \\ r_1^{*\beta_1} & r_1^{*\beta_2} & r_1^{*(1-n)} \\ \dfrac{r_1^{*(\beta_1+1)} - r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)} - r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{r_1^{*(2-n)} - r_0^{*(2-n)}}{2-n} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_0^{*(3-n)} \\ -Qr_1^{*(3-n)} \\ \dfrac{Q}{4-n}\left(r_0^{*(4-n)} - r_1^{*(4-n)}\right) \end{pmatrix} \qquad (73)$$

$$B_4 = -\frac{Q}{4-n}(r_0^*)^{4-n} - \frac{B_1}{1+\beta_1}(r_0^*)^{1+\beta_1} - \frac{B_2}{1+\beta_2}(r_0^*)^{1+\beta_2} - \frac{B_3}{2-n}(r_0^*)^{2-n}\ . \qquad (74)$$

Clamped edge r_0 – Simply supported edge r_1 $[\, w^*(r_0^*) = 0,\ \partial_r w^*(r_0^*) = 0,\ w^*(r_1^*) = 0,\ M^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & r_0^{*(1-n)} \\ \dfrac{r_1^{*(\beta_1+1)} - r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)} - r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{r_1^{*(2-n)} - r_0^{*(2-n)}}{2-n} \\ (\beta_1+\nu)r_1^{*(n+\beta_1+1)} & (\beta_2+\nu)r_1^{*(n+\beta_2+1)} & (1+\nu-n) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_0^{*(3-n)} \\ \dfrac{Q}{4-n}\left(r_0^{*(4-n)} - r_1^{*(4-n)}\right) \\ (n-\nu-3)Qr_1^{*2} \end{pmatrix} \qquad (75)$$

B_4 is given by (74).

Clamped edge r_0 – Free edge r_1 $[\, w^*(r_0^*) = 0, \partial_r w^*(r_0^*) = 0, M^*(r_1^*) = 0, V^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & r_0^{*(1-n)} \\ (\beta_1+\nu)r_1^{*(n+\beta_1+1)} & (\beta_2+\nu)r_1^{*(n+\beta_2+1)} & (1+\nu-n) \\ \tilde{\beta}_1 r_1^{*(n+\beta_1-2)} & \tilde{\beta}_2 r_1^{*(n+\beta_2-2)} & -n\dfrac{1-\nu}{r_1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_0^{*(3-n)} \\ (n-\nu-3)Qr_1^{*2} \\ [n(3-\nu)-8]Qr_1^{*} \end{pmatrix} \tag{76}$$

B_4 is given by (74).

Simply supported edge r_0 *– Clamped edge* r_1 $[\, w^*(r_0^*) = 0\,, M^*(r_0^*) = 0\,, w^*(r_1^*) = 0\,,\ \partial_r w^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} r_1^{*\beta_1} & r_1^{*\beta_2} & r_1^{*(1-n)} \\ \dfrac{r_1^{*(\beta_1+1)} - r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)} - r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{r_1^{*(2-n)} - r_0^{*(2-n)}}{2-n} \\ (\beta_1+\nu)r_0^{*(n+\beta_1+1)} & (\beta_2+\nu)r_0^{*(+\beta_2+1)} & (1+\nu-n) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_1^{*(3-n)} \\ \dfrac{Q}{4-n}\left(r_0^{*(4-n)} - r_1^{*(4-n)}\right) \\ (n-\nu-3)Qr_0^{*2} \end{pmatrix} \tag{77}$$

B_4 is given by (74).

Simply supported edge r_0 *– Simply supported edge* r_1 $[\, w^*(r_0^*) = 0\,, M^*(r_0^*) = 0\,, w^*(r_1^*) = 0\,, M^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} \dfrac{r_1^{*(\beta_1+1)} - r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)} - r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{r_1^{*(2-n)} - r_0^{*(2-n)}}{2-n} \\ (\beta_1+\nu)r_0^{*(n+\beta_1+1)} & (\beta_2+\nu)r_0^{*(n+\beta_2+1)} & (1+\nu-n) \\ (\beta_1+\nu)r_1^{*(n+\beta_1+1)} & (\beta_2+\nu)r_1^{*(n+\beta_2+1)} & (1+\nu-n) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \dfrac{Q}{4-n}\left(r_0^{*(4-n)} - r_1^{*(4-n)}\right) \\ (n-\nu-3)Qr_0^{*2} \\ (n-\nu-3)Qr_1^{*2} \end{pmatrix} \tag{78}$$

B_4 is given by (74).

Simply supported edge r_0 *– Free edge* r_1 $[\, w^*(r_0^*) = 0\,, M^*(r_0^*) = 0\,, M^*(r_1^*) = 0\,, V^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(n+\beta_1+1)} & (\beta_2+\nu)r_0^{*(n+\beta_2+1)} & (1+\nu-n) \\ (\beta_1+\nu)r_1^{*(n+\beta_1+1)} & (\beta_2+\nu)r_1^{*(n+\beta_2+1)} & (1+\nu-n) \\ \tilde{\beta}_1 r_1^{*(n+\beta_1-2)} & \tilde{\beta}_2 r_1^{*(n+\beta_2-2)} & -n\dfrac{1-\nu}{r_1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} (n-\nu-3)Qr_0^{*2} \\ (n-\nu-3)Qr_1^{*2} \\ [n(3-\nu)-8]Qr_1^{*} \end{pmatrix} \tag{79}$$

B_4 is given by (74).

Free edge r_0 *– Clamped edge* r_1 $[\, M^*(r_0^*) = 0\,, V^*(r_0^*) = 0\,, w^*(r_1^*) = 0\,, \partial_r w^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} r_1^{*\beta_1} & r_1^{*\beta_2} & r_1^{*(1-n)} \\ (\beta_1+\nu)r_0^{*(n+\beta_1+1)} & (\beta_2+\nu)r_0^{*(n+\beta_2+1)} & (1+\nu-n) \\ \tilde{\beta}_1 r_0^{*(n+\beta_1-2)} & \tilde{\beta}_2 r_0^{*(n+\beta_2-2)} & -n\dfrac{1-\nu}{r_0} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_1^{*(3-n)} \\ (n-\nu-3)Qr_0^{*2} \\ \left[n(3-\nu)-8\right]Qr_0^* \end{pmatrix} \tag{80}$$

$$B_4 = -\frac{Q}{4-n}r_1^{*(4-n)} - \frac{B_1}{1+\beta_1}r_1^{*(1+\beta_1)} - \frac{B_2}{1+\beta_2}r_1^{*(1+\beta_2)} - \frac{B_3}{2-n}r_1^{*(2-n)} \tag{81}$$

Free edge r_0 – Simply supported edge r_1 $[M^*(r_0^*)=0, V^*(r_0^*)=0, w^*(r_1^*)=0, M^*(r_1^*)=0]$:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(n+\beta_1+1)} & (\beta_2+\nu)r_0^{*(n+\beta_2+1)} & (1+\nu-n) \\ (\beta_1+\nu)r_1^{*(n+\beta_1+1)} & (\beta_2+\nu)r_1^{*(n+\beta_2+1)} & (1+\nu-n) \\ \tilde{\beta}_1 r_0^{*(n+\beta_1-2)} & \tilde{\beta}_2 r_0^{*(n+\beta_2-2)} & -n\dfrac{1-\nu}{r_0} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} (n-\nu-3)Qr_0^{*2} \\ (n-\nu-3)Qr_1^{*2} \\ \left[n(3-\nu)-8\right]Qr_0^* \end{pmatrix} \tag{82}$$

B_4 is given by (81).

(ii) $\underline{n=2}$:

Since in this case $\beta_{1(2)} = -1 \pm \sqrt{2(1-\nu)}$, we conclude that $\beta_{1(2)} \neq -1$, because $\nu \in [-1, 1/2]$. Then, in view of (70) – (71), we have

$$\begin{aligned}
&w^*(r^*) = \frac{Q}{2}r^{*2} + \frac{B_1}{\beta_1+1}r^{*(\beta_1+1)} + \frac{B_2}{\beta_2+1}r^{*(\beta_2+1)} + B_3\ln r^* + B_4 \\
&\partial_r w^*(r^*) = Qr^* + B_1 r^{*\beta_1} + B_2 r^{*\beta_2} + B_3/r^*, \qquad \partial_r^2 w^*(r^*) = Q + \beta_1 B_1 r^{*(\beta_1-1)} + \beta_2 B_2 r^{*(\beta_2-1)} - B_3/r^{*2} \\
&m^*(r^*) = -\frac{1}{r_0^{*2}}\left[2Qr^{*2} + (1+\beta_1)B_1 r^{*(\beta_1+1)} + (1+\beta_2)B_2 r^{*(\beta_2+1)}\right] \\
&\partial_r m^*(r^*) = -\frac{1}{r_0^2}\left[4Qr^* + (1+\beta_1)^2 B_1 r^{*\beta_1} + (1+\beta_2)^2 B_2 r^{*\beta_2}\right] \\
&M^*(r^*) = -\frac{1}{r_0^{*2}}\left[(1+\nu)Qr^{*2} + (\beta_1+\nu)B_1 r^{*(\beta_1+1)} + (\beta_2+\nu)B_2 r^{*(\beta_2+1)} + B_3(\nu-1)\right], \\
&V^*(r^*) = \mp\frac{1}{r_0^{*2}}\left[2(1+\nu)Qr^* + \tilde{\beta}_1 B_1 r^{*\beta_1} + \tilde{\beta}_2 B_2 r^{*\beta_2} - B_3\frac{2(1-\nu)}{r^*}\right]
\end{aligned} \tag{83}$$

with $\tilde{\beta}_i = (1+\beta_i)^2 - 2(1-\nu)$.

The calculation of the integration constants for particular boundary conditions is described in what follows.

Clamped edge r_0 *– Clamped edge* r_1 $[\, w^*(r_0^*)=0\,,\partial_r w^*(r_0^*)=0\,,\ w^*(r_1^*)=0\,,\ \partial_r w^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & 1/r_0^* \\ r_1^{*\beta_1} & r_1^{*\beta_2} & 1/r_1^* \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \ln\left(r_1^*/r_0^*\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_0^* \\ -Qr_1^* \\ \dfrac{Q}{2}\left(r_0^{*2}-r_1^{*2}\right) \end{pmatrix} \tag{84}$$

$$B_4 = -\frac{Q}{2}r_0^{*2} - \frac{B_1}{1+\beta_1}r_0^{*(1+\beta_1)} - \frac{B_2}{1+\beta_2}r_0^{*(1+\beta_2)} - B_3\ln r_0^* \tag{85}$$

Clamped edge r_0 *– Simply supported edge* r_1 $[\, w^*(r_0^*)=0\,,\partial_r w^*(r_0^*)=0\,, w^*(r_1^*)=0\,, M^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & 1/r_0^* \\ (\beta_1+\nu)r_1^{*(\beta_1+1)} & (\beta_2+\nu)r_1^{*(\beta_2+1)} & \nu-1 \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \ln\left(r_1^*/r_0^*\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_0^* \\ -(1+\nu)Qr_1^{*2} \\ \dfrac{Q}{2}\left(r_0^{*2}-r_1^{*2}\right) \end{pmatrix} \tag{86}$$

B_4 is given by (85).

Clamped edge r_0 *– Free edge* r_1 $[\, w^*(r_0^*)=0\,,\partial_r w^*(r_0^*)=0\,, M^*(r_1^*)=0\,, V^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & 1/r_0^* \\ (\beta_1+\nu)r_1^{*(\beta_1+1)} & (\beta_2+\nu)r_1^{*(\beta_2+1)} & \nu-1 \\ \tilde{\beta}_1 r_1^{*\beta_1} & \tilde{\beta}_2 r_1^{*\beta_2} & -\dfrac{2(1-\nu)}{r_1^*} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_0^* \\ -(1+\nu)Qr_1^{*2} \\ -2(1+\nu)Qr_1^* \end{pmatrix} \tag{87}$$

B_4 is given by (85).

Simply supported edge r_0 *– Clamped edge* r_1 $[\, w^*(r_0^*)=0\,, M^*(r_0^*)=0\,, w^*(r_1^*)=0\,,\partial_r w^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_1^{*\beta_1} & r_1^{*\beta_2} & 1/r_1^* \\ (\beta_1+\nu)r_0^{*(\beta_1+1)} & (\beta_2+\nu)r_0^{*(\beta_2+1)} & \nu-1 \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \ln\left(r_1^*/r_0^*\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Qr_1^* \\ -(1+\nu)Qr_0^{*2} \\ \dfrac{Q}{2}\left(r_0^{*2}-r_1^{*2}\right) \end{pmatrix} \tag{88}$$

B_4 is given by (85).

Simply supported edge r_0 – Simply supported edge r_1 [$w^*(r_0^*)=0$, $M^*(r_0^*)=0$, $w^*(r_1^*)=0$, $M^*(r_1^*)=0$]:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+1)} & (\beta_2+\nu)r_0^{*(\beta_2+1)} & \nu-1 \\ (\beta_1+\nu)r_1^{*(\beta_1+1)} & (\beta_2+\nu)r_1^{*(\beta_2+1)} & \nu-1 \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \ln\left(r_1^*/r_0^*\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(1+\nu)Qr_0^{*2} \\ -(1+\nu)Qr_1^{*2} \\ \dfrac{Q}{2}\left(r_0^{*2}-r_1^{*2}\right) \end{pmatrix} \tag{89}$$

B_4 is given by (85).

Simply supported edge r_0 – Free edge r_1 [$w^*(r_0^*)=0$, $M^*(r_0^*)=0$, $M^*(r_1^*)=0$, $V^*(r_1^*)=0$]:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+1)} & (\beta_2+\nu)r_0^{*(\beta_2+1)} & \nu-1 \\ (\beta_1+\nu)r_1^{*(\beta_1+1)} & (\beta_2+\nu)r_1^{*(\beta_2+1)} & \nu-1 \\ \tilde{\beta}_1 r_1^{*\beta_1} & \tilde{\beta}_2 r_1^{*\beta_2} & -\dfrac{2(1-\nu)}{r_1^*} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(1+\nu)Qr_0^{*2} \\ -(1+\nu)Qr_1^{*2} \\ -2(1+\nu)Qr_1^{*} \end{pmatrix} \tag{90}$$

B_4 is given by (85).

Free edge r_0 – Clamped edge r_1 [$M^*(r_0^*)=0$, $V^*(r_0^*)=0$, $w^*(r_1^*)=0$, $\partial_r w^*(r_1^*)=0$]:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+1)} & (\beta_2+\nu)r_0^{*(\beta_2+1)} & \nu-1 \\ \tilde{\beta}_1 r_0^{*\beta_1} & \tilde{\beta}_2 r_0^{*\beta_2} & -\dfrac{2(1-\nu)}{r_0^*} \\ r_1^{*\beta_1} & r_1^{*\beta_2} & 1/r_1^* \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(1+\nu)Qr_0^{*2} \\ -2(1+\nu)Qr_0^{*} \\ -Qr_1^{*} \end{pmatrix} \tag{91}$$

$$B_4 = -\frac{Q}{2}r_1^{*2} - \frac{B_1}{1+\beta_1}r_1^{*(1+\beta_1)} - \frac{B_2}{1+\beta_2}r_1^{*(1+\beta_2)} - B_3\ln r_1^* \tag{92}$$

Free edge r_0 – Simply supported edge r_1 [$M^*(r_0^*)=0$, $V^*(r_0^*)=0$, $w^*(r_1^*)=0$, $M^*(r_1^*)=0$]:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+1)} & (\beta_2+\nu)r_0^{*(\beta_2+1)} & \nu-1 \\ \tilde{\beta}_1 r_0^{*\beta_1} & \tilde{\beta}_2 r_0^{*\beta_2} & -\dfrac{2(1-\nu)}{r_0^*} \\ (\beta_1+\nu)r_1^{*(\beta_1+1)} & (\beta_2+\nu)r_1^{*(\beta_2+1)} & \nu-1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(1+\nu)Qr_0^{*2} \\ -2(1+\nu)Qr_0^{*} \\ -(1+\nu)Qr_1^{*2} \end{pmatrix} \tag{93}$$

B_4 is given by (92).

(iii) $\underline{n=4}$:

Since in this case $\beta_{1(2)} = -2 \pm \sqrt{5-4\nu}$, we conclude that $\beta_{1(2)} \neq -1$, because $\nu \in [-1, 1/2]$. Then, in view of (70) – (71), we have

$$
\begin{aligned}
& w^*(r^*) = Q \ln r^* + \frac{B_1}{\beta_1 + 1} r^{*(\beta_1+1)} + \frac{B_2}{\beta_2 + 1} r^{*(\beta_2+1)} - \frac{B_3}{2r^{*2}} + B_4 \\
& \partial_r w^*(r^*) = Q / r^* + B_1 r^{*\beta_1} + B_2 r^{*\beta_2} + B_3 / r^{*3} \ , \\
& \partial_r^2 w^*(r^*) = -Q / r^{*2} + \beta_1 B_1 r^{*(\beta_1-1)} + \beta_2 B_2 r^{*(\beta_2-1)} - 3B_3 / r^{*4} \\
& m^*(r^*) = -\frac{1}{r_0^{*4}} \left[(1+\beta_1) B_1 r^{*(\beta_1+3)} + (1+\beta_2) B_2 r^{*(\beta_2+3)} - 2B_3 \right] \\
& \partial_r m^*(r^*) = -\frac{1}{r_0^{*4}} \left[(1+\beta_1)(3+\beta_1) B_1 r^{*(\beta_1+2)} + (1+\beta_2)(3+\beta_2) B_2 r^{*(\beta_2+2)} \right] \\
& M^*(r^*) = -\frac{1}{r_0^{*4}} \left[(\nu-1) Q r^{*2} + (\beta_1+\nu) B_1 r^{*(\beta_1+3)} + (\beta_2+\nu) B_2 r^{*(\beta_2+3)} + (\nu-3) B_3 \right], \\
& V^*(r^*) = \mp \frac{1}{r_0^{*4}} \left[-4(1-\nu) Q r^* + \tilde{\beta}_1 B_1 r^{*(\beta_1+2)} + \tilde{\beta}_2 B_2 r^{*(\beta_2+2)} - B_3 \frac{4(1-\nu)}{r^*} \right]
\end{aligned}
\tag{94}
$$

with $\tilde{\beta}_i = (1+\beta_i)(3+\beta_i) - 4(1-\nu)$.

The calculation of the integration constants for particular boundary conditions is described in what follows.

Clamped edge r_0 – Clamped edge r_1 [$w^*(r_0^*) = 0$, $\partial_r w^*(r_0^*) = 0$, $w^*(r_1^*) = 0$, $\partial_r w^*(r_1^*) = 0$]:

$$
\begin{pmatrix}
r_0^{*\beta_1} & r_0^{*\beta_2} & 1/r_0^{*3} \\
r_1^{*\beta_1} & r_1^{*\beta_2} & 1/r_1^{*3} \\
\dfrac{r_1^{*(\beta_1+1)} - r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)} - r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{1}{2}\left(\dfrac{1}{r_0^{*2}} - \dfrac{1}{r_1^{*2}}\right)
\end{pmatrix}
\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}
=
\begin{pmatrix} -Q / r_0^* \\ -Q / r_1^* \\ Q \ln\left(r_0^* / r_1^*\right) \end{pmatrix}
\tag{95}
$$

$$
B_4 = -Q \ln r_0^* - \frac{B_1}{1+\beta_1} r_0^{*(1+\beta_1)} - \frac{B_2}{1+\beta_2} r_0^{*(1+\beta_2)} + \frac{B_3}{2r_0^{*2}}
\tag{96}
$$

Clamped edge r_0 – Simply supported edge r_1 $[\, w^*(r_0^*)=0\,, M^*(r_0^*)=0\,,\ w^*(r_1^*)=0\,,\ \partial_r w^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & 1/r_0^{*3} \\ (\beta_1+\nu)r_1^{*(\beta_1+3)} & (\beta_2+\nu)r_1^{*(\beta_2+3)} & \nu-3 \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{1}{2}\left(\dfrac{1}{r_0^{*2}}-\dfrac{1}{r_1^{*2}}\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Q/r_0^* \\ (1-\nu)Qr_1^{*2} \\ Q\ln\left(r_0^*/r_1^*\right) \end{pmatrix} \tag{97}$$

B_4 is given by (96).

Clamped edge r_0 – Free edge r_1 $[\, w^*(r_0^*)=0\,, \partial_r w^*(r_0^*)=0\,, M^*(r_1^*)=0\,, V^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_0^{*\beta_1} & r_0^{*\beta_2} & 1/r_0^{*3} \\ (\beta_1+\nu)r_1^{*(\beta_1+3)} & (\beta_2+\nu)r_1^{*(\beta_2+3)} & \nu-3 \\ \tilde{\beta}_1 r_1^{*(\beta_1+2)} & \tilde{\beta}_2 r_1^{*(\beta_2+2)} & -\dfrac{4(1-\nu)}{r_1^*} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Q/r_0^* \\ (1-\nu)Qr_1^{*2} \\ 4(1-\nu)Qr_1^* \end{pmatrix} \tag{98}$$

B_4 is given by (96).

Simply supported edge r_0 – Clamped edge r_1 $[\, w^*(r_0^*)=0\,, M^*(r_0^*)=0\,,\ w^*(r_1^*)=0\,,\ \partial_r w^*(r_1^*)=0\,]$:

$$\begin{pmatrix} r_1^{*\beta_1} & r_1^{*\beta_2} & 1/r_1^{*3} \\ (\beta_1+\nu)r_0^{*(\beta_1+3)} & (\beta_2+\nu)r_0^{*(\beta_2+3)} & \nu-3 \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{1}{2}\left(\dfrac{1}{r_0^{*2}}-\dfrac{1}{r_1^{*2}}\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -Q/r_1^* \\ (1-\nu)Qr_0^{*2} \\ Q\ln\left(r_0^*/r_1^*\right) \end{pmatrix} \tag{99}$$

B_4 is given by (96).

Simply supported edge r_0 – Simply supported edge r_1 $[\, w^*(r_0^*)=0\,, M^*(r_0^*)=0\,, w^*(r_1^*)=0\,, M^*(r_1^*)=0\,]$:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+3)} & (\beta_2+\nu)r_0^{*(\beta_2+3)} & \nu-3 \\ (\beta_1+\nu)r_1^{*(\beta_1+3)} & (\beta_2+\nu)r_1^{*(\beta_2+3)} & \nu-3 \\ \dfrac{r_1^{*(\beta_1+1)}-r_0^{*(\beta_1+1)}}{\beta_1+1} & \dfrac{r_1^{*(\beta_2+1)}-r_0^{*(\beta_2+1)}}{\beta_2+1} & \dfrac{1}{2}\left(\dfrac{1}{r_0^{*2}}-\dfrac{1}{r_1^{*2}}\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} (1-\nu)Qr_0^{*2} \\ (1-\nu)Qr_1^{*2} \\ Q\ln\left(r_0^*/r_1^*\right) \end{pmatrix} \tag{100}$$

B_4 is given by (96).

Simply supported edge r_0 – Free edge r_1 $[\, w^*(r_0^*)=0\,, M^*(r_0^*)=0\,, M^*(r_1^*)=0\,, V^*(r_1^*)=0\,]$:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+3)} & (\beta_2+\nu)r_0^{*(\beta_2+3)} & \nu-3 \\ (\beta_1+\nu)r_1^{*(\beta_1+3)} & (\beta_2+\nu)r_1^{*(\beta_2+3)} & \nu-3 \\ \tilde{\beta}_1 r_1^{*(\beta_1+2)} & \tilde{\beta}_2 r_1^{*(\beta_2+2)} & -\dfrac{4(1-\nu)}{r_1^*} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} (1-\nu)Qr_0^{*2} \\ (1-\nu)Qr_1^{*2} \\ 4(1-\nu)Qr_1^* \end{pmatrix} \tag{101}$$

B_4 is given by (96).

Free edge r_0 – Clamped edge r_1 $[\, M^*(r_0^*)=0\,, V^*(r_0^*)=0\,, w^*(r_1^*)=0\,,\ \partial_r w^*(r_1^*)=0\,]$:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+3)} & (\beta_2+\nu)r_0^{*(\beta_2+3)} & \nu-3 \\ \tilde{\beta}_1 r_0^{*(\beta_1+2)} & \tilde{\beta}_2 r_0^{*(\beta_2+2)} & -\dfrac{4(1-\nu)}{r_0^*} \\ r_1^{*\beta_1} & r_1^{*\beta_2} & 1/r_1^{*3} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} (1-\nu)Qr_0^{*2} \\ 4(1-\nu)Qr_0^* \\ -Q/r_1^* \end{pmatrix} \tag{102}$$

$$B_4 = -Q\ln r_1^* - \frac{B_1}{1+\beta_1} r_1^{*(1+\beta_1)} - \frac{B_2}{1+\beta_2} r_1^{*(1+\beta_2)} + \frac{B_3}{2r_1^{*2}} \tag{103}$$

Free edge r_0 – Simply supported edge r_1 $[\, M^*(r_0^*)=0\,, V^*(r_0^*)=0\,, w^*(r_1^*)=0\,, M^*(r_1^*)=0\,]$:

$$\begin{pmatrix} (\beta_1+\nu)r_0^{*(\beta_1+3)} & (\beta_2+\nu)r_0^{*(\beta_2+3)} & \nu-3 \\ (\beta_1+\nu)r_1^{*(\beta_1+3)} & (\beta_2+\nu)r_1^{*(\beta_2+3)} & \nu-3 \\ \tilde{\beta}_1 r_0^{*(\beta_1+2)} & \tilde{\beta}_2 r_0^{*(\beta_2+2)} & -\dfrac{4(1-\nu)}{r_0^*} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} (1-\nu)Qr_0^{*2} \\ (1-\nu)Qr_1^{*2} \\ 4(1-\nu)Qr_0^* \end{pmatrix} \tag{104}$$

B_4 is given by (103).

6.2 Plate with constant bending stiffness

In this section, we summarize the exact solutions for the circular plate with constant bending stiffness $D^*(r) = D(r)/D_0 \equiv 1$. This is necessary, because in previous derivations we have excluded the case with $n = 0$. Now, instead of the governing equation (54), we have

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \nabla^2 w^*}{\partial r^*} \right) = 1\,. \tag{105}$$

Integrating Eq. (105) we arrive at

$$w^*(r^*)=\frac{r^{*4}}{64}+B_1\frac{r^{*2}}{4}\left(\ln r^*-1\right)+B_2\frac{r^{*2}}{4}+B_3\ln r^*+B_4 \tag{106}$$

where the integration constants B_1, B_2, B_3, B_4 are determined by the prescribed boundary conditions. For this purpose, we need to derive the expressions for the bending moment and the generalized shear force. Thus, in view of (92), we receive

$$\begin{aligned}
&\partial_r w^*(r^*)=\frac{r^{*3}}{16}+B_1\frac{r^*}{2}\left(\ln r^*-\frac{1}{2}\right)+B_2\frac{r^*}{2}+\frac{B_3}{r^*}, \quad \partial_r^2 w^*(r^*)=\frac{3r^{*2}}{16}+\frac{B_1}{2}\left(\ln r^*+\frac{1}{2}\right)+\frac{B_2}{2}-\frac{B_3}{r^{*2}}\\
&m^*(r^*)=-\nabla^2 w(r^*)=-\left(\partial_r^2 w^*(r^*)+\frac{1}{r^*}\partial_r w^*(r^*)\right)=-\left[\frac{r^{*2}}{4}+B_1\ln r^*+B_2\right]\\
&M^*(r^*)=m^*(r^*)+\frac{1-\nu}{r^*}\partial_r w^*(r^*)=-\left[\frac{3+\nu}{16}r^{*2}+B_1\left(\frac{1+\nu}{2}\ln r^*+\frac{1-\nu}{4}\right)+\frac{1+\nu}{2}B_2+\frac{\nu-1}{r^{*2}}B_3\right]\\
&V^*(r^*)=\partial_r m^*(r^*)=-\left(\frac{r^*}{2}+\frac{B_1}{r^*}\right).
\end{aligned} \tag{107}$$

(A) *Plate with central circular hole*

For the sake of brevity, we introduce the notations

$$\rho_a:=\frac{r_a^*}{2}\left(\ln r_a^*-\frac{1}{2}\right),\qquad R_a:=\frac{r_a^{*2}}{4}\left(\ln r_a^*-1\right),\qquad \mu_a:=\frac{1+\nu}{2}\ln r_a^*+\frac{1-\nu}{4}\ \text{ with } (a=0,1).$$

Clamped edge r_0 – Clamped edge r_1 $[\,w^*(r_0^*)=0,\ \partial_r w^*(r_0^*)=0,\ w^*(r_1^*)=0,\ \partial_r w^*(r_1^*)=0\,]$:

$$\begin{pmatrix}\rho_0 & r_0^*/2 & 1/r_0^*\\ \rho_1 & r_1^*/2 & 1/r_1^*\\ R_1-R_0 & \left(r_1^{*2}-r_0^{*2}\right)/4 & \ln\left(r_1^*/r_0^*\right)\end{pmatrix}\begin{pmatrix}B_1\\ B_2\\ B_3\end{pmatrix}=\begin{pmatrix}-r_0^{*3}/16\\ -r_1^{*3}/16\\ \left(r_0^{*4}-r_1^{*4}\right)/64\end{pmatrix}, \tag{108}$$

$$B_4=-\frac{r_0^{*4}}{64}-B_1R_0-B_2\frac{r_0^{*2}}{4}-B_3\ln r_0^* \tag{109}$$

Clamped edge r_0 – Simply supported edge r_1 $[\,w^*(r_0^*)=0,\ \partial_r w^*(r_0^*)=0,\ w^*(r_1^*)=0,\ M^*(r_1^*)=0\,]$:

$$\begin{pmatrix}\rho_0 & r_0^*/2 & 1/r_0^*\\ \mu_1 & (\nu+1)/2 & (\nu-1)/r_1^{*2}\\ R_1-R_0 & \left(r_1^{*2}-r_0^{*2}\right)/4 & \ln\left(r_1^*/r_0^*\right)\end{pmatrix}\begin{pmatrix}B_1\\ B_2\\ B_3\end{pmatrix}=\begin{pmatrix}-r_0^{*3}/16\\ -(\nu+3)r_1^{*2}/16\\ \left(r_0^{*4}-r_1^{*4}\right)/64\end{pmatrix}, \tag{110}$$

B_4 is given by (109).

Clamped edge r_0 *– Free edge* r_1 $[\, w^*(r_0^*) = 0\,, \partial_r w^*(r_0^*) = 0\,, M^*(r_1^*) = 0\,, V^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} \rho_0 & r_0^*/2 & 1/r_0^* \\ \mu_1 & (\nu+1)/2 & (\nu-1)/r_1^{*2} \\ 1/r_1^* & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -r_0^{*3}/16 \\ -(\nu+3)r_1^{*2}/16 \\ -r_1^*/2 \end{pmatrix}, \tag{111}$$

B_4 is given by (109).

Simply supported edge r_0 *– Clamped edge* r_1 $[\, w^*(r_0^*) = 0\,, M^*(r_0^*) = 0\,, w^*(r_1^*) = 0\,,\ \partial_r w^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} \rho_1 & r_1^*/2 & 1/r_1^* \\ \mu_0 & (\nu+1)/2 & (\nu-1)/r_0^{*2} \\ R_1 - R_0 & \left(r_1^{*2} - r_0^{*2}\right)/4 & \ln\left(r_1^*/r_0^*\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -r_1^{*3}/16 \\ -(\nu+3)r_0^{*2}/16 \\ \left(r_0^{*4} - r_1^{*4}\right)/64 \end{pmatrix}, \tag{112}$$

B_4 is given by (109).

Simply supported edge r_0 *– Simply supported edge* r_1 $[\, w^*(r_0^*) = 0\,, M^*(r_0^*) = 0\,, w^*(r_1^*) = 0\,, M^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} \mu_0 & (\nu+1)/2 & (\nu-1)/r_0^{*2} \\ \mu_1 & (\nu+1)/2 & (\nu-1)/r_1^{*2} \\ R_1 - R_0 & \left(r_1^{*2} - r_0^{*2}\right)/4 & \ln\left(r_1^*/r_0^*\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(\nu+3)r_0^{*2}/16 \\ -(\nu+3)r_1^{*2}/16 \\ \left(r_0^{*4} - r_1^{*4}\right)/64 \end{pmatrix}, \tag{113}$$

B_4 is given by (109).

Simply supported edge r_0 *– Free edge* r_1

$$\begin{pmatrix} \mu_0 & (\nu+1)/2 & (\nu-1)/r_0^{*2} \\ \mu_1 & (\nu+1)/2 & (\nu-1)/r_1^{*2} \\ 1/r_1^* & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(\nu+3)r_0^{*2}/16 \\ -(\nu+3)r_1^{*2}/16 \\ -r_1^*/2 \end{pmatrix}, \tag{100}$$

B_4 is given by (109).

Free edge r_0 *– Clamped edge* r_1 $[\, M^*(r_0^*) = 0\,, V^*(r_0^*) = 0\,, w^*(r_1^*) = 0\,,\ \partial_r w^*(r_1^*) = 0\,]$:

$$\begin{pmatrix} \mu_0 & (\nu+1)/2 & (\nu-1)/r_0^{*2} \\ \rho_1 & r_1^*/2 & 1/r_1^* \\ 1/r_0^* & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(\nu+3)r_0^{*2}/16 \\ -r_1^{*3}/16 \\ -r_0^*/2 \end{pmatrix}, \tag{114}$$

$$B_4 = -\frac{r_1^{*4}}{64} - B_1 R_1 - B_2 \frac{r_1^{*2}}{4} - B_3 \ln r_1^* \tag{115}$$

Free edge r_0 – Simply supported edge r_1 $[M^*(r_0^*)=0, V^*(r_0^*)=0, w^*(r_1^*)=0, M^*(r_1^*)=0]$:

$$\begin{pmatrix} \mu_0 & (\nu+1)/2 & (\nu-1)/r_0^{*2} \\ \mu_1 & (\nu+1)/2 & (\nu-1)/r_1^{*2} \\ 1/r_0^* & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} -(\nu+3)r_0^{*2}/16 \\ -(\nu+3)r_1^{*2}/16 \\ -r_0^*/2 \end{pmatrix}, \tag{116}$$

B_4 is given by (115).

(B) *Plate without any hole*

Owing to completeness, we present also exact solutions for the circular plate without any hole. Because of the angular symmetry we require $\partial_r w^*(r^* = 0) = 0$. Hence and from (107), we need to put $B_3 \equiv 0$. Furthermore, in view of Eq. (107), the requirement of finite value $M^*(r^* = 0)$ results into the requirement $B_1 \equiv 0$. Thus, for circular plate with $r^* \in [0, r_1^*]$, the exact solution is determined by two integration constants as

$$w^*(r^*) = \frac{r^{*4}}{64} + B_2 \frac{r^{*2}}{4} + B_4, \quad \partial_r w^*(r^*) = \frac{r^{*3}}{16} + B_2 \frac{r^*}{2}, \quad \partial_r^2 w^*(r^*) = \frac{3r^{*2}}{16} + \frac{B_2}{2}$$

$$m^*(r^*) = -\nabla^2 w^*(r^*) = -\left(\frac{r^{*2}}{4} + B_2\right) \tag{117}$$

$$M^*(r^*) = m^*(r^*) + \frac{1-\nu}{r^*}\partial_r w^*(r^*) = -\left(\frac{3+\nu}{16} r^{*2} + \frac{1+\nu}{2} B_2\right) \quad , \quad V^*(r^*) = \partial_r m^*(r^*) = -\frac{r^*}{2} .$$

Clamped edge r_1 $[w^*(r_1^*)=0, \ \partial_r w^*(r_1^*)=0]$:

$$B_2 = -\frac{r_1^{*2}}{8}, \quad B_4 = \frac{r_1^{*4}}{64} . \tag{118}$$

Simply supported edge r_1 $[w^*(r_1^*)=0, M^*(r_1^*)=0]$:

$$B_2 = -\frac{3+\nu}{8(1+\nu)} r_1^{*2} \quad , \quad B_4 = \frac{5+\nu}{1+\nu}\frac{r_1^{*4}}{64} . \tag{119}$$

6.3 Plate with transversally graded Young modulus

Let us assume the power-law gradation of the Young modulus as shown in Sect. 4.1 and $E_H = 1$, $h^* = 1$. Then, $D_H^* = 1$, $D^* = D_V^*$, and the governing equations (59) – (61) can be rewritten as

$$D^*\nabla^2 w^* + m^* = 0 \tag{120}$$

$$\nabla^2 m^* = -\lambda^* \ , \qquad \lambda^* := \frac{H}{(1-\nu)F}\lambda_0^* \tag{121}$$

$$\partial_r^2 u_r^* + \frac{1}{r^*}\partial_r u_r^* - \frac{1}{r^{*2}}u_r^* + \frac{\gamma_p}{D_V^*}\partial_r m^* = 0 \ . \tag{122}$$

One can solve easily this system of ODE with the result

$$\begin{aligned}
m^*(r^*) &= -\frac{\lambda^*}{4}r^{*2} - b_1 D^* \ln r^* - b_2 D^* \\
w^*(r^*) &= \frac{\lambda^*}{64D^*}r^{*4} + b_1\frac{r^{*2}}{4}\left(\ln r^* - 1\right) + b_2\frac{r^{*2}}{4} + b_3 \ln r^* + b_4 \\
u_r^*(r) &= \frac{\gamma_p\lambda^*}{16D^*}r^3 + \frac{\gamma_p b_1}{2} r\ln r + b_5 r + b_6\frac{1}{r}, \\
M^* &= \frac{1-\nu}{H}\beta_p\left(\frac{F}{D^*}m^* + \frac{H}{r}\partial_r w^*\right) + \frac{1-\nu}{H}\alpha_p\left(F\partial_r + \frac{\nu}{r}\right)u_r^* \\
V^* &= \pm\frac{1-\nu}{H}F\partial_r m^*
\end{aligned} \tag{123}$$

where b_1, …, b_6 are the integration constants which are determined by prescribed boundary conditions. For simplicity, we omitted superstar in the radial variable though it is dimensionless according to definitions introduced above.

Clamped edge r_0 - Clamped edge r_1 $[\ w^*(r_0^*) = 0\ , \partial_r w^*(r_0^*) = 0,\ w^*(r_1^*) = 0\ ,\ \partial_r w^*(r_1^*) = 0,\ u_r^*(r_0) = 0\ ,\ u_r^*(r_1) = 0\ ]$:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$$A_{ik}B_k = P_i\ , \quad (i,k = 1,2,3)\ , \text{ with } \ B_k = b_k$$

$$A_{11} = \rho_0\,, \quad A_{12} = r_0^*/2\,, \quad A_{13} = 1/r_0^*\,, \qquad P_1 = -\lambda^* r_0^{*3}/16D^*$$

$$A_{21} = \rho_1\,, \quad A_{22} = r_1^*/2\,, \quad A_{23} = 1/r_1^*\,, \qquad P_2 = -\lambda^* r_1^{*3}/16D^*$$

$$A_{31} = R_1 - R_0\,, \quad A_{32} = \left(r_1^{*2} - r_0^{*2}\right)/4\,, \quad A_{33} = \ln\left(r_1^*/r_0^*\right), \qquad P_3 = \lambda^*\left(r_0^{*4} - r_1^{*4}\right)/64D^*$$

where, we have introduced the notations

$$\rho_a := \frac{r_a^*}{2}\left(\ln r_a^* - \frac{1}{2}\right), \qquad R_a := \frac{r_a^{*2}}{4}\left(\ln r_a^* - 1\right), \text{ with } (a = 0,1). \tag{124}$$

Subsequently b_4, b_5, b_3 are given by

$$b_4 = -\frac{\lambda^*}{64D^*}(r_0^*)^4 - b_1 R_0 - b_2\left(\frac{r_0^*}{2}\right)^2 - b_3 \ln r_0^* \ ,$$

$$b_5 = -\frac{\lambda^* \gamma_p}{16D^* r_{01}}\left(\frac{(r_0^*)^3}{r_1^*} - \frac{(r_1^*)^3}{r_0^*}\right) + b_1 \frac{1}{2r_{01}}\left(\frac{r_1^*}{r_0^*}\ln r_1^* - \frac{r_0^*}{r_1^*}\ln r_0^*\right), \qquad r_{01} := \frac{r_0^*}{r_1^*} - \frac{r_1^*}{r_0^*} \tag{125}$$

$$b_6 = \frac{\lambda^* \gamma_p}{16D^* r_{01}} r_0^* r_1^* \left((r_0^*)^2 - (r_1^*)^2\right) - b_1 \frac{\gamma_p}{2r_{01}} r_0^* r_1^* \ln\left(r_1^* / r_0^*\right)$$

Clamped edge r_0 - *Simply supported edge* r_1 $[\, w^*(r_0^*) = 0\,, \partial_r w^*(r_0^*) = 0\,,\ w^*(r_1^*) = 0\,,\ M^*(r_1^*) = 0\,,\ T_r^*(r_1) = 0\,]$:

The integration constants b_1, b_5, b_6 are evaluated by solving the system of equations

$$A_{ik} B_k = P_i\ , \quad (i,k = 1,2,3)\,,\ \text{with}\ \ B_1 = b_1\,,\ B_2 = b_5\,,\ B_3 = b_6$$

$$A_{11} = \left(\frac{\alpha_p \gamma_p}{2\beta_p}(\nu + F) + \frac{H}{2} - F\right)\ln r_1^* - a_1\left(\frac{H}{2} - F\right) + \frac{\alpha_p \gamma_p}{2\beta_p}F - \frac{H}{4} - \frac{a_3 H}{(r_1^*)^2}\,,$$

$$A_{12} = (\nu + F)\frac{\alpha_p}{\beta_p}, \quad A_{13} = (\nu - F)\frac{\alpha_p}{\beta_p (r_1^*)^2}\,,$$

$$P_1 = -\frac{\lambda^* r_1^{*2}}{16D^*}\left(\frac{\alpha_p \gamma_p}{\beta_p}(\nu + 3F) + H - 4F\right) - (F - \frac{H}{2})a_0 - \frac{H}{r_1^{*2}}a_2$$

$$A_{21} = \frac{\gamma_p}{2} r_0 \ln r_0\,, \quad A_{22} = r_0^*, \quad A_{23} = 1/r_0^*, \qquad P_2 = -\lambda^* \gamma_p r_0^{*3} / 16D^*$$

$$A_{31} = \gamma_p\left[\left(F - \frac{H}{2}\right)\left(\frac{1}{2} + a_1\right) - \frac{a_3 H}{r_1^{*2}}\right], \quad A_{32} = \nu + F\ , \quad A_{33} = \frac{\nu - F}{r_1^{*2}}\,, \qquad P_3 = \gamma_p\left[\left(\frac{H}{2} - F\right)a_0 - \frac{H}{r_1^{*2}}a_2\right]$$

where, we have introduced the notations

$$a_1' = \left(\frac{r_1^{*2}}{2}\right)^2 - \left(\frac{r_0^{*2}}{2}\right)^2\left(1 + 2\ln\frac{r_1^*}{r_0^*}\right), \quad a_0 = \frac{\lambda^*}{64D^*}\frac{r_1^{*4} - r_0^{*4}\left(1 + 4\ln(r_1^*/)\right)}{a_1'}\,,$$

$$a_1 = \left(R_1 - R_0 - \frac{r_0^{*2}}{2}\ln\frac{r_1^*}{r_0^*}(\ln r_0^* - 0.5)\right)/\,a_1', \quad a_2 = -\frac{\lambda^*}{16D^*}r_0^{*4} + \frac{r_0^{*2}}{2}a_0\,,$$

$$a_3 = \frac{r_0^{*2}}{2}\left(\ln r_0^* - 0.5 - a_1\right)$$

The rest unknown $b_\cdot$ are given by

$b_2 = -a, - a_1 b_1, b_3 = a_2 - a_3 b_1, b_4$ by (125).

Clamped edge r_0 - *Free edge* r_1 $[\, w^*(r_0^*)=0\,, \partial_r w^*(r_0^*)=0\,,\ M^*(r_1^*)=0\,,\ V^*(r_1^*)=0\,,\ u_r^*(r_0)=0\,,\ T_r^*(r_1)=0\,]$:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$$A_{ik}B_k = P_i\ , \quad (i,k=1,2,3)\,,\ \text{with}\ \ B_k = b_k\ ,$$

$$A_{11}=1\,,\quad A_{12}=0\,,\quad A_{13}=0\,,\qquad P_1 = -\frac{\lambda^*}{2D^*} r_1^{*2}$$

$$A_{21}=0\,,\quad A_{22} = \frac{H}{2} - F + \frac{\alpha_p}{\beta_p}(\nu+F)a_2 - \frac{\alpha_p}{\beta_p}\left(\frac{r_0^*}{r_1^*}\right)^2 (\nu-F)a_2\,,$$

$$A_{23} = \frac{H}{r_1^{*2}} + \frac{\alpha_p}{\beta_p}(\nu+F)a_3 - \frac{\alpha_p}{\beta_p}\left(\frac{r_0^*}{r_1^*}\right)^2 (\nu-F)a_3\,,$$

$$P_2 = -\frac{\lambda^*}{16D^*} r_1^{*2}\left(\frac{\alpha_p\gamma_p}{\beta_p}(3F+\nu)+H-4F\right) - b_1\left[\left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F)+\frac{H}{2}-F\right)\ln r_1^* - \frac{H}{4} + \frac{\alpha_p\gamma_p}{2\beta_p}F\right] -$$
$$-\frac{\alpha_p}{\beta_p}(\nu+F)a_4 + \frac{\alpha_p}{\beta_p r_1^{*2}}(\nu-F)\left(r_0^{*2}a_4 - a_0\right),$$

$$A_{31}=0\,,\quad A_{32} = r_0^*/2\,,\quad A_{33} = 1/r_0^*\,,\quad P_3 = -\frac{\lambda^*}{2D^*} r_0^{*3} - b_1\rho_0$$

where

$$a_0 = -\frac{\lambda^*\gamma_p}{16D^*} r_0^{*4} - b_1\frac{\gamma_p}{2} r_0^{*2}\ln r_o^*\ ,\qquad a_1' = \nu + F - \left(\frac{r_0^*}{r_1^*}\right)^2(\nu-F)\ ,\quad a_2 = \frac{\gamma_p}{a_1'}(F - \frac{H}{2})\ ,$$

$$a_3 = -\frac{\gamma_p H}{a_1' r_1^{*2}}\ ,\quad a_4 = \frac{1}{a_1'}\left\{\frac{F-\nu}{r_1^{*2}}a_0 + b_1\frac{\gamma_p}{2}\left(\frac{H}{2}-F\right)\right\}$$

The rest unknown $b_\cdot$ are given by

$b_5 = b_2 a_2 + b_3 a_3 + a_4\,,\quad b_6 = -r_0^{*2} b_5 + a_0\,,\quad b_4$ by (125).

Simply supported edge r_0 - *Clamped edge* r_1 $[\, w^*(r_0^*)=0\,, M^*(r_0^*)=0\,,\ w^*(r_1^*)=0\,,\ \partial_r w^*(r_1^*)=0\,,\ u_r^*(r_1)=0\,]$:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$$A_{ik}B_k = P_i\ , \quad (i,k=1,2,3)\ , \text{ with } \ B_k = b_k\ ,$$

$$A_{11} = R_1 - R_0\ , \quad A_{12} = \left(r_1^{*2} - r_0^{*2}\right)/4\ , \quad A_{13} = \ln\left(\frac{r_1^*}{r_0^*}\right), \qquad P_1 = \frac{\lambda^*}{64D^*}\left(r_0^{*4} - r_1^{*4}\right)$$

$$A_{21} = \rho_1\ , \quad A_{22} = r_1^*/2\ , \quad A_{22} = 1/r_1^*\ , \quad P_2 = -\frac{\lambda^*}{16D^*}r_1^{*3}$$

$$A_{31} = \left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F) + \frac{H}{2} - F\right)\ln r_0^* + \frac{\alpha_p\gamma_p}{2\beta_p}F - \frac{H}{4} + \frac{\alpha_p}{\beta_p}(\nu+F)a_1 + \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_5 - a_1 r_1^{*2}\right)$$

$$A_{32} = \frac{H}{2} - F + \frac{\alpha_p}{\beta_p}(\nu+F)a_2 - \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_2\ , \quad A_{33} = \frac{H}{r_0^{*2}} + \frac{\alpha_p}{\beta_p}(\nu+F)a_3 - \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_3$$

$$P_3 = -\frac{\lambda^* r_0^{*2}}{16D^*}\left(\frac{\alpha_p\gamma_p}{\beta_p}(\nu+3F) + H - 4F\right) - \frac{\alpha_p}{\beta_p}(\nu+F)a_6 - \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_0 - a_6 r_1^{*2}\right)$$

where

$$a_0 = -\frac{\lambda^*\gamma_p}{16D^*}r_1^{*4}\ , \quad a_1 = -\frac{1}{a_4'}\left[\frac{\gamma_p}{2}\left(F - \frac{H}{2}\right) + \frac{(\nu-F)a_5}{r_0^{*2}}\right], \quad a_2 = \frac{\gamma_p}{a_4'}\left(F - \frac{H}{2}\right)\ , \quad a_3 = -\frac{\gamma_p H}{a_4' r_0^{*2}}\ ,$$

$$a_5 = -\frac{\gamma_p}{2}r_1^{*2}\ln r_1^*\ , \quad a_6 = -\frac{\nu-F}{r_0^{*2}}\frac{a_0}{a_4'}\ , \quad a_4' = \nu + F - \frac{\nu-F}{r_0^{*2}}r_1^{*2}\ .$$

The rest unknown b. are given by

$b_5 = b_1a_1 + b_2a_2 + b_3a_3 + a_6\ , \quad b_6 = b_1a_5 - r_1^{*2}b_5 + a_0\ , \quad b_4$ by (125).

Simply supported edge r_0 *- Simply supported edge* r_1 [$w^*(r_0^*) = 0\ , M^*(r_0^*) = 0\ ,\ w^*(r_1^*) = 0\ ,\ M^*(r_1^*) = 0\ ,$ $T_r^*(r_0) = 0\ ,\ u_r^*(r_1) = 0$]:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$A_{ik}B_k = Pi$, $(i,k = 1,2,3)$, with $B_k = b_k$,

$$A_{11} = R_1 - R_0\ , \quad A_{12} = \left(r_1^{*2} - r_0^{*2}\right)/4\ , \quad A_{13} = \ln\left(\frac{r_1^*}{r_0^*}\right), \qquad P_1 = \frac{\lambda^*}{64D^*}\left(r_0^{*4} - r_1^{*4}\right)$$

$$A_{21} = \left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F) + \frac{H}{2} - F\right)\ln r_0^* + \frac{\alpha_p\gamma_p}{2\beta_p}F - \frac{H}{4} + \frac{\alpha_p}{\beta_p}(\nu+F)a_1 + \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_5 - a_1 r_1^{*2}\right),$$

$$A_{22}=\frac{H}{2}-F+\frac{\alpha_p}{\beta_p}(\nu+F)a_2-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_2\,,\quad A_{23}=\frac{H}{r_0^{*2}}+\frac{\alpha_p}{\beta_p}(\nu+F)a_3-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_3\,,$$

$$P_2=-\frac{\lambda^* r_0^{*2}}{16D^*}\left(\frac{\alpha_p\gamma_p}{\beta_p}(\nu+3F)+H-4F\right)-\frac{\alpha_p}{\beta_p}(\nu+F)a_6-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_0-a_6 r_1^{*2}\right),$$

$$A_{31}=\left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F)+\frac{H}{2}-F\right)\ln r_1^*+\frac{\alpha_p\gamma_p}{2\beta_p}F-\frac{H}{4}+\frac{\alpha_p}{\beta_p}(\nu+F)a_1+\frac{\alpha_p}{\beta_p r_1^{*2}}(\nu-F)\left(a_5-a_1 r_1^{*2}\right),$$

$$A_{32}=\frac{H}{2}-F+2\frac{\alpha_p}{\beta_p}Fa_2\,,\quad A_{33}=\frac{H}{r_1^{*2}}+2\frac{\alpha_p}{\beta_p}Fa_3\,,$$

$$P_3=-\frac{\lambda^* r_1^{*2}}{16D^*}\left(\frac{\alpha_p\gamma_p}{\beta_p}(\nu+3F)+H-4F\right)-\frac{\alpha_p}{\beta_p}(\nu+F)a_6-\frac{\alpha_p}{\beta_p r_1^{*2}}(\nu-F)\left(a_0-a_6 r_1^{*2}\right),$$

where

$$a_0=-\frac{\lambda^*\gamma_p}{16D^*}r_1^{*4}\,,\quad a_1=-\frac{1}{a_4'}\left[\frac{\gamma_p}{2}\left(F-\frac{H}{2}\right)+\frac{(\nu-F)a_5}{r_0^{*2}}\right],\quad a_2=\frac{\gamma_p}{a_4'}\left(F-\frac{H}{2}\right)\,,\quad a_3=-\frac{\gamma_p H}{a_4' r_0^{*2}}\,,$$

$$a_5=-\frac{\gamma_p}{2}r_1^{*2}\ln r_1^*\,,\quad a_6=-\frac{\nu-F}{r_0^{*2}}\frac{a_0}{a_4'}\,,\quad a_4'=\nu+F-\frac{\nu-F}{r_0^{*2}}r_1^{*2}\,.$$

The rest unknown $b_.$ are given by

$$b_5=b_1a_1+b_2a_2+b_3a_3+a_6\,,\quad b_6=b_1a_5-r_1^{*2}b_5+a_0\,,\quad b_4 \text{ by (125).}$$

Simply supported edge r_0 - Free edge r_1 $[\,w^*(r_0^*)=0\,,M^*(r_0^*)=0\,,\ M^*(r_1^*)=0\,,\ V^*(r_1^*)=0\,,\ T_r^*(r_0)=0\,,\ u_r^*(r_1)=0\,]$:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$$A_{ik}B_k=P_i\,,\quad (i,k=1,2,3)\,,\ \text{with}\ B_k=b_k,$$

$$A_{11}=1,\quad A_{12}=0\,,\quad A_{13}=0\,,\qquad P_1=-\frac{\lambda^*}{2D^*}r_1^{*2}$$

$$A_{21}=\left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F)+\frac{H}{2}-F\right)\ln r_0^*+\frac{\alpha_p\gamma_p}{2\beta_p}F-\frac{H}{4}+\frac{\alpha_p}{\beta_p}(\nu+F)a_1+\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_5-a_1 r_1^{*2}\right),$$

$$A_{22}=\frac{H}{2}-F+\frac{\alpha_p}{\beta_p}(\nu+F)a_2-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_2\,,\quad A_{23}=\frac{H}{r_0^{*2}}+\frac{\alpha_p}{\beta_p}(\nu+F)a_3-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_3\,,$$

$$P_2=-\frac{\lambda^* r_0^{*2}}{16D^*}\left(\frac{\alpha_p\gamma_p}{\beta_p}(\nu+3F)+H-4F\right)-\frac{\alpha_p}{\beta_p}(\nu+F)a_6-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_0-a_6 r_1^{*2}\right),$$

$$A_{31}=\left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F)+\frac{H}{2}-F\right)\ln r_1^*+\frac{\alpha_p\gamma_p}{2\beta_p}F-\frac{H}{4}+\frac{\alpha_p}{\beta_p}(\nu+F)a_1+\frac{\alpha_p}{\beta_p r_1^{*2}}(\nu-F)\left(a_5-a_1 r_1^{*2}\right),$$

$$A_{32}=\frac{H}{2}-F+2\frac{\alpha_p}{\beta_p}Fa_2,\quad A_{33}=\frac{H}{r_1^{*2}}+2\frac{\alpha_p}{\beta_p}Fa_3,$$

$$P_3=-\frac{\lambda^* r_1^{*2}}{16D^*}\left(\frac{\alpha_p\gamma_p}{\beta_p}(\nu+3F)+H-4F\right)-\frac{\alpha_p}{\beta_p}(\nu+F)a_6-\frac{\alpha_p}{\beta_p r_1^{*2}}(\nu-F)\left(a_0-a_6 r_1^{*2}\right),$$

where

$$a_0=-\frac{\lambda^*\gamma_p}{16D^*}r_1^{*4},\quad a_1=-\frac{1}{a_4'}\left[\frac{\gamma_p}{2}\left(F-\frac{H}{2}\right)+\frac{(\nu-F)a_5}{r_0^{*2}}\right],\quad a_2=\frac{\gamma_p}{a_4'}\left(F-\frac{H}{2}\right),\quad a_3=-\frac{\gamma_p H}{a_4' r_0^{*2}},$$

$$a_5=-\frac{\gamma_p}{2}r_1^{*2}\ln r_1^*,\quad a_6=-\frac{\nu-F}{r_0^{*2}}\frac{a_0}{a_4'},\quad a_4'=\nu+F-\frac{\nu-F}{r_0^{*2}}r_1^{*2}.$$

The rest unknown b. are given by

$$b_5=b_1a_1+b_2a_2+b_3a_3+a_6,\quad b_6=b_1a_5-r_1^{*2}b_5+a_0,\quad b_4 \text{ by (125).}$$

Free edge r_0 - *Clamped edge* r_1 $[M^*(r_0^*)=0,V^*(r_0^*)=0,\ w^*(r_1^*)=0,\ \partial_r w^*(r_1^*)=0,\ T_r^*(r_0)=0,\ u_r^*(r_1)=0]$:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$$A_{ik}B_k=P_i,\quad (i,k=1,2,3),\ \text{with}\ B_k=b_k,$$

$$A_{11}=1,\quad A_{12}=0,\quad A_{13}=0,\qquad P_1=-\frac{\lambda^*}{2D^*}r_0^{*2}$$

$$A_{21}=\left(\frac{\alpha_p\gamma_p}{2\beta_p}(\nu+F)+\frac{H}{2}-F\right)\ln r_0^*+\frac{\alpha_p\gamma_p}{2\beta_p}F-\frac{H}{4}+\frac{\alpha_p}{\beta_p}(\nu+F)a_1+\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_5-a_1 r_1^{*2}\right),$$

$$A_{22}=\frac{H}{2}-F+\frac{\alpha_p}{\beta_p}(\nu+F)a_2-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_2,$$

$$A_{23}=\frac{H}{r_0^{*2}}+\frac{\alpha_p}{\beta_p}(\nu+F)a_3-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)r_1^{*2}a_3,$$

$$P_2=-\frac{\lambda^* r_0^{*2}}{16D^*}\left(\frac{\alpha_p\gamma_p}{\beta_p}(\nu+3F)+H-4F\right)-\frac{\alpha_p}{\beta_p}(\nu+F)a_6-\frac{\alpha_p}{\beta_p r_0^{*2}}(\nu-F)\left(a_0-a_6 r_1^{*2}\right),$$

$$A_{31}=\rho_1,\quad A_{32}=r_1^*/2,\quad A_{33}=1/r_1^*,\qquad P_3=-\frac{\lambda^*}{16D^*}r_1^{*3}$$

where

$$a_0 = -\frac{\lambda^* \gamma_p}{16D^*} r_1^{*4}, \quad a_1 = -\frac{1}{a_4'}\left[\frac{\gamma_p}{2}\left(F - \frac{H}{2}\right) + \frac{(\nu - F)a_5}{r_0^{*2}}\right], \quad a_2 = \frac{\gamma_p}{a_4'}\left(F - \frac{H}{2}\right), \quad a_3 = -\frac{\gamma_p H}{a_4' r_0^{*2}},$$

$$a_5 = -\frac{\gamma_p}{2} r_1^{*2} \ln r_1^*, \quad a_6 = -\frac{\nu - F}{r_0^{*2}}\frac{a_0}{a_4'}, \quad a_4' = \nu + F - \frac{\nu - F}{r_0^{*2}} r_1^{*2}.$$

The rest unknown b. are given by

$$b_5 = b_1 a_1 + b_2 a_2 + b_3 a_3 + a_6, \quad b_6 = b_1 a_5 - r_1^{*2} b_5 + a_0,$$
$$b_4 = -\frac{\lambda^*}{64D^*}(r_1^*)^4 - b_1 R_1 - b_2\left(\frac{r_1^*}{2}\right)^2 - b_3 \ln r_1^*. \tag{126}$$

Free edge r_0 *- Simply supported edge* r_1 $[\, M^*(r_0^*) = 0, V^*(r_0^*) = 0,\ w^*(r_1^*) = 0,\ M^*(r_1^*) = 0,\ T_r^*(r_0) = 0,\ u_r^*(r_1) = 0\,]$:

The integration constants b_1, b_2, b_3 are evaluated by solving the system of equations

$$A_{ik} B_k = P_i, \quad (i,k = 1,2,3), \text{ with } B_k = b_k,$$

$$A_{11} = 1, \quad A_{12} = 0, \quad A_{13} = 0, \qquad P_1 = -\frac{\lambda^*}{2D^*} r_0^{*2}$$

$$A_{21} = \left(\frac{\alpha_p \gamma_p}{2\beta_p}(\nu + F) + \frac{H}{2} - F\right)\ln r_0^* + \frac{\alpha_p \gamma_p}{2\beta_p}F - \frac{H}{4} + \frac{\alpha_p}{\beta_p}(\nu + F)a_1 + \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu - F)\left(a_5 - a_1 r_1^{*2}\right),$$

$$A_{22} = \frac{H}{2} - F + \frac{\alpha_p}{\beta_p}(\nu + F)a_2 - \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu - F) r_1^{*2} a_2, \quad A_{23} = \frac{H}{r_0^{*2}} + \frac{\alpha_p}{\beta_p}(\nu + F)a_3 - \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu - F) r_1^{*2} a_3,$$

$$P_2 = -\frac{\lambda^* r_0^{*2}}{16D^*}\left(\frac{\alpha_p \gamma_p}{\beta_p}(\nu + 3F) + H - 4F\right) - \frac{\alpha_p}{\beta_p}(\nu + F)a_6 - \frac{\alpha_p}{\beta_p r_0^{*2}}(\nu - F)\left(a_0 - a_6 r_1^{*2}\right),$$

$$A_{31} = \left(\frac{\alpha_p \gamma_p}{2\beta_p}(\nu + F) + \frac{H}{2} - F\right)\ln r_1^* + \frac{\alpha_p \gamma_p}{2\beta_p}F - \frac{H}{4} + \frac{\alpha_p}{\beta_p}(\nu + F)a_1 + \frac{\alpha_p}{\beta_p r_1^{*2}}(\nu - F)\left(a_5 - a_1 r_1^{*2}\right),$$

$$A_{32} = \frac{H}{2} - F + 2\frac{\alpha_p}{\beta_p}F a_2, \quad A_{33} = \frac{H}{r_1^{*2}} + 2\frac{\alpha_p}{\beta_p}F a_3,$$

$$P_3 = -\frac{\lambda^* r_1^{*2}}{16D^*}\left(\frac{\alpha_p \gamma_p}{\beta_p}(\nu + 3F) + H - 4F\right) - \frac{\alpha_p}{\beta_p}(\nu + F)a_6 - \frac{\alpha_p}{\beta_p r_1^{*2}}(\nu - F)\left(a_0 - a_6 r_1^{*2}\right),$$

where

$$a_0 = -\frac{\lambda^* \gamma_p}{16D^*} r_1^{*4}, \quad a_1 = -\frac{1}{a_4'}\left[\frac{\gamma_p}{2}\left(F - \frac{H}{2}\right) + \frac{(\nu - F)a_5}{r_0^{*2}}\right], \quad a_2 = \frac{\gamma_p}{a_4'}\left(F - \frac{H}{2}\right), \quad a_3 = -\frac{\gamma_p H}{a_4' r_0^{*2}},$$

$$a_5 = -\frac{\gamma_p}{2} r_1^{*2} \ln r_1^*, \quad a_6 = -\frac{\nu - F}{r_0^{*2}} \frac{a_0}{a_4'}, \quad a_4' = \nu + F - \frac{\nu - F}{r_0^{*2}} r_1^{*2}.$$

The rest unknown *b*. are given by

$b_5 = b_1 a_1 + b_2 a_2 + b_3 a_3 + a_6$, $\quad b_6 = b_1 a_5 - r_1^{*2} b_5 + a_0$, $\quad b_4$ by (126).

7. Conclusions

In this paper, we presented a systematic derivation of the governing equations for the plate bending with assuming the Kirchhoff hypothesis and allowing the spatial variation of the bending stiffness. It is shown that even within the Kirchhoff – Love theory the coupling between the in-plane and deflection deformations cannot be omitted, if the Young modulus is variable across the plate thickness. On the other hand, the classical Kirchhoff – Love theory without coupling can be applied to bending of FGM plates with continuous in-plane variation of both the material coefficients and plate thickness. Owing to low accuracy of approximation of higher order derivatives, their appearance gives rise to difficulties in the development of numerical computational methods. In order to exclude high order derivatives from the original biharmonic problem, we propose to utilize the presented decomposed formulation for coupled two field variables governed by the second order differential equations. In the expressions for the relevant boundary quantities the derivatives of the employed field variables do not exceed the first order. The dimensionless formulation is developed in order to enhance the applicability of numerical results. In order to test the numerical method proposed in Part II of this paper, we derived the exact solutions for circular plates with variable bending stiffness according to power-law gradation in radial direction. For the sake of completeness, we presented also the exact solutions for boundary value problems in plates with constant bending stiffness. The bending problem for the FGM plates with the transversally graded Young modulus is formally also the problem with a constant bending stiffness, but it is impossible to get the solution for such a FGM plate from the solution of a homogeneous plate by a proper specification of the bending stiffness. The coupling between the deflections and in-plane deformations plays an important role. On the other hand, the solution for a homogeneous plate can be obtained from the solution for the FGM plate with transversal gradation of the Young modulus by proper selection of material coefficients. The most important consequence of the mentioned coupling is the fact that utilization of the plane stress formulation is questionable in FGM plates with transversally graded Young's modulus, though in bending of homogeneous plates and/or plates with in-plane gradation of the bending stiffness the application of the plane stress formulation is transparent.

Acknowledgements

The financial support by APVV-0032-10 (Slovak Research and Development Agency) is greatly acknowledged.

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